Checking the Brumer-Stark conjecture using PARI/GP

Xavier-François Roblot

IGD, Université Claude Bernard – Lyon 1

September 16th, 2004

Statement of the conjecture

- Definitions
- The Brumer element
- The Brumer-Stark Conjecture
- 2 Current status of the conjecture
 - Some reductions and special cases
 - Further results
- 3 Checking the conjecture on an example
 - The example
 - The strategy
 - The verification

Definitions The Brumer element The Brumer-Stark Conjecture

• k is a number field of degree n

- K is a finite abelian extension over k
- $G := \operatorname{Gal}(K/k)$
- w_K is the number of roots of unity in K
- Cl_K is the class group of K
- *S* is the set of the infinite primes of *k* and of the finite prime ideals in *k* that ramify in *K*
- For each $\sigma \in G$, the partial zeta-function is

$$\zeta_{\mathcal{S}}(s,\sigma) := \sum_{\substack{(\mathfrak{a}, \mathcal{S}) = 1 \\ \sigma_{\mathfrak{a}} = \sigma}} \frac{1}{\mathcal{N}\mathfrak{a}^{s}}$$

イロン 不同と 不同と 不同と

Definitions The Brumer element The Brumer-Stark Conjecture

- k is a number field of degree n
- K is a finite abelian extension over k
- $G := \operatorname{Gal}(K/k)$
- w_K is the number of roots of unity in K
- Cl_K is the class group of K
- S is the set of the infinite primes of k and of the finite prime ideals in k that ramify in K
- For each $\sigma \in G$, the partial zeta-function is

$$\zeta_{\mathcal{S}}(s,\sigma) := \sum_{\substack{(\mathfrak{a}, \mathcal{S})=1\\\sigma_{\mathfrak{a}} = \sigma}} \frac{1}{\mathcal{N}\mathfrak{a}^{s}}$$

イロン イヨン イヨン イヨン

Definitions The Brumer element The Brumer-Stark Conjecture

- k is a number field of degree n
- K is a finite abelian extension over k
- $G := \operatorname{Gal}(K/k)$
- w_K is the number of roots of unity in K
- Cl_{K} is the class group of K
- S is the set of the infinite primes of k and of the finite prime ideals in k that ramify in K
- For each $\sigma \in G$, the partial zeta-function is

$$\zeta_{\mathcal{S}}(s,\sigma) := \sum_{\substack{(\mathfrak{a},S)=1\\\sigma_{\mathfrak{a}}=\sigma}} \frac{1}{\mathcal{N}\mathfrak{a}^{s}}$$

Definitions The Brumer element The Brumer-Stark Conjecture

- k is a number field of degree n
- K is a finite abelian extension over k
- $G := \operatorname{Gal}(K/k)$
- w_K is the number of roots of unity in K
- $\operatorname{Cl}_{\mathcal{K}}$ is the class group of \mathcal{K}
- S is the set of the infinite primes of k and of the finite prime ideals in k that ramify in K

• For each $\sigma \in G$, the partial zeta-function is

$$\zeta_{\mathcal{S}}(s,\sigma) := \sum_{\substack{(\mathfrak{a}, \mathcal{S})=1\\\sigma_{\mathfrak{a}}=\sigma}} \frac{1}{\mathcal{N}\mathfrak{a}^{s}}$$

Definitions The Brumer element The Brumer-Stark Conjecture

- k is a number field of degree n
- K is a finite abelian extension over k
- $G := \operatorname{Gal}(K/k)$
- w_K is the number of roots of unity in K
- Cl_K is the class group of K
- S is the set of the infinite primes of k and of the finite prime ideals in k that ramify in K
- For each $\sigma \in G$, the partial zeta-function is

$$\zeta_{\mathcal{S}}(s,\sigma) := \sum_{\substack{(\mathfrak{a}, \mathcal{S}) = 1 \ \sigma_{\mathfrak{a}} = \sigma}} rac{1}{\mathcal{N}\mathfrak{a}^s}$$

Definitions The Brumer element The Brumer-Stark Conjecture

Theorem (Deligne and Ribet, Barsky, and Pi. Cassou-Noguès) For every $\sigma \in G$ $w_K \zeta_S(0, \sigma) \in \mathbb{Z}$

The *Brumer element* is the element of the group ring $\mathbb{Z}[G]$ defined by

$$\gamma := w_K \sum_{\sigma \in G} \zeta_S(0, \sigma) \sigma^{-1}$$

イロン イヨン イヨン イヨン

Definitions The Brumer element The Brumer-Stark Conjecture



The *Brumer element* is the element of the group ring $\mathbb{Z}[G]$ defined by

$$\gamma := w_K \sum_{\sigma \in G} \zeta_S(0, \sigma) \sigma^{-1}$$

イロン イヨン イヨン イヨン

Definitions The Brumer element The Brumer-Stark Conjecture

The Brumer-Stark Conjecture

Conjecture (The Brumer part)

The element γ kills $Cl_{\mathcal{K}}$.

That is, for every fractional ideal $\mathfrak A$ of K, the ideal $\mathfrak A^\gamma$ is principal.

Let K° be the set of *anti-units* of K

$$K^{\circ} := \{ x \in K : |x|_{\mathfrak{P}_{\infty}} = 1, \ \forall \mathfrak{P}_{\infty} \mid \infty \}$$

Conjecture (The Stark part)

For every fractional ideal \mathfrak{A} of K, there exists a generator $\alpha_{\mathfrak{A}}$ of \mathfrak{A}^{γ} that is an anti-unit. Furthermore, define $\lambda_{\mathfrak{A}} \in \overline{K}$ by $\lambda_{\mathfrak{A}}^{W_{K}} = \alpha_{\mathfrak{A}}$, then $K(\lambda_{\mathfrak{A}})/k$ is an abelian extension.

(日) (同) (E) (E) (E)

Definitions The Brumer element The Brumer-Stark Conjecture

The Brumer-Stark Conjecture

Conjecture (The Brumer part)

The element γ kills $Cl_{\mathcal{K}}$.

That is, for every fractional ideal \mathfrak{A} of K, the ideal \mathfrak{A}^{γ} is principal.

Let K° be the set of *anti-units* of K

$$K^{\circ} := \{ x \in K : |x|_{\mathfrak{P}_{\infty}} = 1, \ \forall \mathfrak{P}_{\infty} \mid \infty \}$$

Conjecture (The Stark part)

For every fractional ideal \mathfrak{A} of K, there exists a generator $\alpha_{\mathfrak{A}}$ of \mathfrak{A}^{γ} that is an anti-unit. Furthermore, define $\lambda_{\mathfrak{A}} \in \overline{K}$ by $\lambda_{\mathfrak{A}}^{W_{\mathcal{K}}} = \alpha_{\mathfrak{A}}$, then $K(\lambda_{\mathfrak{A}})/k$ is an abelian extension.

ヘロン 人間と 人間と 人間と

Definitions The Brumer element The Brumer-Stark Conjecture

The Brumer-Stark Conjecture

Conjecture (The Brumer part)

The element γ kills $Cl_{\mathcal{K}}$.

That is, for every fractional ideal \mathfrak{A} of K, the ideal \mathfrak{A}^{γ} is principal.

Let K° be the set of *anti-units* of K

$${\mathcal K}^\circ := \{x \in {\mathcal K}: |x|_{{\mathfrak P}_\infty} = 1, \; orall {\mathfrak P}_\infty \mid \infty \}$$

Conjecture (The Stark part)

For every fractional ideal \mathfrak{A} of K, there exists a generator $\alpha_{\mathfrak{A}}$ of \mathfrak{A}^{γ} that is an anti-unit. Furthermore, define $\lambda_{\mathfrak{A}} \in \overline{K}$ by $\lambda_{\mathfrak{A}}^{W_{K}} = \alpha_{\mathfrak{A}}$, then $K(\lambda_{\mathfrak{A}})/k$ is an abelian extension.

ヘロン 人間と 人間と 人間と

3

Definitions The Brumer element The Brumer-Stark Conjecture

The Brumer-Stark Conjecture

Conjecture (The Brumer part)

The element γ kills $Cl_{\mathcal{K}}$.

That is, for every fractional ideal \mathfrak{A} of K, the ideal \mathfrak{A}^{γ} is principal.

Let K° be the set of *anti-units* of K

$${\mathcal K}^\circ := \{x \in {\mathcal K}: |x|_{{\mathfrak P}_\infty} = 1, \; orall {\mathfrak P}_\infty \mid \infty\}$$

Conjecture (The Stark part)

For every fractional ideal \mathfrak{A} of K, there exists a generator $\alpha_{\mathfrak{A}}$ of \mathfrak{A}^{γ} that is an anti-unit. Furthermore, define $\lambda_{\mathfrak{A}} \in \overline{K}$ by $\lambda_{\mathfrak{A}}^{W_{\kappa}} = \alpha_{\mathfrak{A}}$, then $K(\lambda_{\mathfrak{A}})/k$ is an abelian extension.

イロン イヨン イヨン イヨン

э

• The conjecture is true if $k = \mathbb{Q}$ (Stickelberger's Theorem)

- The conjecture is true if k is not totally real or K is not totally complex
- The conjecture is satisfied for ${\mathfrak A}$ if it is a principal ideal
- The conjecture is true if K is principal
- The set of ideals satisfying the conjecture forms a group, stable under the action of *G*

イロン イヨン イヨン イヨン

- The conjecture is true if $k = \mathbb{Q}$ (Stickelberger's Theorem)
- The conjecture is true if k is not totally real or K is not totally complex
- The conjecture is satisfied for $\mathfrak A$ if it is a principal ideal
- The conjecture is true if K is principal
- The set of ideals satisfying the conjecture forms a group, stable under the action of *G*

イロン イヨン イヨン イヨン

- The conjecture is true if $k = \mathbb{Q}$ (Stickelberger's Theorem)
- The conjecture is true if k is not totally real or K is not totally complex
- \bullet The conjecture is satisfied for ${\mathfrak A}$ if it is a principal ideal
- The conjecture is true if *K* is principal
- The set of ideals satisfying the conjecture forms a group, stable under the action of *G*

- The conjecture is true if $k = \mathbb{Q}$ (Stickelberger's Theorem)
- The conjecture is true if k is not totally real or K is not totally complex
- \bullet The conjecture is satisfied for ${\mathfrak A}$ if it is a principal ideal
- The conjecture is true if K is principal
- The set of ideals satisfying the conjecture forms a group, stable under the action of *G*

- The conjecture is true if $k = \mathbb{Q}$ (Stickelberger's Theorem)
- The conjecture is true if k is not totally real or K is not totally complex
- \bullet The conjecture is satisfied for ${\mathfrak A}$ if it is a principal ideal
- The conjecture is true if K is principal
- The set of ideals satisfying the conjecture forms a group, stable under the action of *G*

- if K/k is quadratic [Tate]
- if G ≃ Z/2Z × Z/2Z in general, and when G is of exponent 2 and has order > 4, assuming K/k is a tame extension [Sands]
- if |G| = 4 and K/k is a sub-extension of a non-abelian Galois extension K/k_0 of degree 8 [Tate]
- if K/k is a sub-extension of an abelian Galois extension K/k₀ for which the conjecture is true [Sands, Hayes]
- if $G \simeq \mathbb{Z}/4\mathbb{Z}$ and k is real quadratic [Greither]
- if [K : k] = 6, and [k : Q] = 2, or 3 and the discriminant of k is coprime with 6 (except for some very special cases) [Greither-Roblot-Tangedal]

- if *K*/*k* is quadratic [Tate]
- if G ≃ Z/2Z × Z/2Z in general, and when G is of exponent 2 and has order > 4, assuming K/k is a tame extension [Sands]
- if |G| = 4 and K/k is a sub-extension of a non-abelian Galois extension K/k_0 of degree 8 [Tate]
- if K/k is a sub-extension of an abelian Galois extension K/k₀ for which the conjecture is true [Sands, Hayes]
- if $G \simeq \mathbb{Z}/4\mathbb{Z}$ and k is real quadratic [Greither]
- if [K : k] = 6, and [k : Q] = 2, or 3 and the discriminant of k is coprime with 6 (except for some very special cases) [Greither-Roblot-Tangedal]

- if K/k is quadratic [Tate]
- if G ≃ Z/2Z × Z/2Z in general, and when G is of exponent 2 and has order > 4, assuming K/k is a tame extension [Sands]
- if |G| = 4 and K/k is a sub-extension of a non-abelian Galois extension K/k_0 of degree 8 [Tate]
- if K/k is a sub-extension of an abelian Galois extension K/k_0 for which the conjecture is true [Sands, Hayes]
- if $G \simeq \mathbb{Z}/4\mathbb{Z}$ and k is real quadratic [Greither]
- if [K : k] = 6, and [k : ℚ] = 2, or 3 and the discriminant of k is coprime with 6 (except for some very special cases) [Greither-Roblot-Tangedal]

- if K/k is quadratic [Tate]
- if G ≃ Z/2Z × Z/2Z in general, and when G is of exponent 2 and has order > 4, assuming K/k is a tame extension [Sands]
- if |G| = 4 and K/k is a sub-extension of a non-abelian Galois extension K/k_0 of degree 8 [Tate]
- if K/k is a sub-extension of an abelian Galois extension K/k_0 for which the conjecture is true [Sands, Hayes]
- if $G \simeq \mathbb{Z}/4\mathbb{Z}$ and k is real quadratic [Greither]
- if [K : k] = 6, and [k : ℚ] = 2, or 3 and the discriminant of k is coprime with 6 (except for some very special cases) [Greither-Roblot-Tangedal]

・ロト ・回ト ・ヨト ・ヨト

- if K/k is quadratic [Tate]
- if G ≃ Z/2Z × Z/2Z in general, and when G is of exponent 2 and has order > 4, assuming K/k is a tame extension [Sands]
- if |G| = 4 and K/k is a sub-extension of a non-abelian Galois extension K/k_0 of degree 8 [Tate]
- if K/k is a sub-extension of an abelian Galois extension K/k₀ for which the conjecture is true [Sands, Hayes]
- if $G \simeq \mathbb{Z}/4\mathbb{Z}$ and k is real quadratic [Greither]
- if [K : k] = 6, and [k : Q] = 2, or 3 and the discriminant of k is coprime with 6 (except for some very special cases)
 [Greither-Roblot-Tangedal]

・ロト ・回ト ・ヨト ・ヨト

3

- if K/k is quadratic [Tate]
- if G ≃ Z/2Z × Z/2Z in general, and when G is of exponent 2 and has order > 4, assuming K/k is a tame extension [Sands]
- if |G| = 4 and K/k is a sub-extension of a non-abelian Galois extension K/k_0 of degree 8 [Tate]
- if K/k is a sub-extension of an abelian Galois extension K/k_0 for which the conjecture is true [Sands, Hayes]
- if $G \simeq \mathbb{Z}/4\mathbb{Z}$ and k is real quadratic [Greither]
- if [K : k] = 6, and [k : ℚ] = 2, or 3 and the discriminant of k is coprime with 6 (except for some very special cases)
 [Greither-Roblot-Tangedal]

Statement of the conjecture	
Current status of the conjecture	
Checking the conjecture on an example	

The example The strategy The verification

Let $k = \mathbb{Q}(\sqrt{69})$, and $K = K^+(j)$ where K^+ is the ray class field of k of conductor 3 and j is a primitive third root of unity. This example is one of the exceptions not covered by [GRT].

The example The strategy The verification

• Compute the Brumer element using *L*-functions

Find a minimal set {𝔄₁,...,𝔄_s} of Z[G]-generators of Cl_K
For each 𝔄

 Compute 32' and check if it is principal Gall (*b* a generator of 30', find a unit *v* such that or :=: v/) is annihilarity.

・ロン ・ 日 ・ ・ 日 ・ ・ 日 ・

Checking the conjectu	re	on a	an example
Current status	of	the	conjecture
Statement	of	the	conjecture

- Compute the Brumer element using *L*-functions
- Find a minimal set $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_s\}$ of $\mathbb{Z}[G]$ -generators of Cl_K

For each ^ຊ

- Compute \mathfrak{A}^{γ} and check if it is principal
- Call β a generator of 20^o, find a unit u such that α := uβ is an anti-unit
- Check if $K(\alpha^{1/m})$ is an abelian extension of k

・ロン ・回 と ・ ヨ と ・ ヨ と

-2

Checking the conjectu	re	on a	an example
Current status	of	the	conjecture
Statement	of	the	conjecture

- Compute the Brumer element using L-functions
- Find a minimal set $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_s\}$ of $\mathbb{Z}[G]$ -generators of Cl_K
- For each ${\mathfrak A}$
 - Compute \mathfrak{A}^γ and check if it is principal
 - Call β a generator of $\mathfrak{A}^{\gamma},$ find a unit u such that $\alpha:=u\beta$ is an anti-unit
 - Check if $K(\alpha^{1/w_{\kappa}})$ is an abelian extension of k

Checking the conjectur	e on a	n example
Current status o	f the	coniecture
Statement of	of the	conjecture

- Compute the Brumer element using L-functions
- Find a minimal set $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_s\}$ of $\mathbb{Z}[G]$ -generators of Cl_K
- For each ${\mathfrak A}$
 - $\bullet\,$ Compute \mathfrak{A}^γ and check if it is principal
 - Call β a generator of $\mathfrak{A}^{\gamma},$ find a unit u such that $\alpha:=u\beta$ is an anti-unit
 - Check if $K(\alpha^{1/w_{\kappa}})$ is an abelian extension of k

Checking the conjectu	re	on a	an example
Current status	of	the	conjecture
Statement	of	the	conjecture

- Compute the Brumer element using L-functions
- Find a minimal set $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_s\}$ of $\mathbb{Z}[G]$ -generators of Cl_K
- For each \mathfrak{A}
 - $\bullet\,$ Compute \mathfrak{A}^γ and check if it is principal
 - Call β a generator of $\mathfrak{A}^\gamma,$ find a unit u such that $\alpha:=u\beta$ is an anti-unit
 - Check if $K(\alpha^{1/w_{\mathcal{K}}})$ is an abelian extension of k

Checking the conjectu	re	on a	an example
Current status	of	the	conjecture
Statement	of	the	conjecture

- Compute the Brumer element using *L*-functions
- Find a minimal set $\{\mathfrak{A}_1, \ldots, \mathfrak{A}_s\}$ of $\mathbb{Z}[G]$ -generators of Cl_K
- For each \mathfrak{A}
 - $\bullet\,$ Compute \mathfrak{A}^γ and check if it is principal
 - Call β a generator of $\mathfrak{A}^\gamma,$ find a unit u such that $\alpha:=u\beta$ is an anti-unit
 - Check if $K(\alpha^{1/w_{\kappa}})$ is an abelian extension of k

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Let's start GP!

・ロト ・回ト ・ヨト ・ヨト

æ

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

$$\gamma = w_K \sum_{\chi \in \hat{G}} \overline{L_S(0,\chi)} e_{\chi}$$
 where $e_{\chi} := rac{1}{|G|} \sum_{\sigma \in G} ar{\chi}(\sigma) \sigma^{-1}$

・ロト・(四ト・(川下・(日下・(日下)

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Let
$$\mathfrak{g}$$
 be a generator of $\operatorname{Cl}_k(3\infty_1\infty_2)$.
Let $\sigma := \sigma_{\mathfrak{g}}$. Thus $G = \langle \sigma \rangle$.
Let $\zeta_6 := \exp(2i\pi/6)$.

The character χ_a represented by [a] is the one defined by

$$\chi_a(\sigma) := \zeta_6^a.$$

An element $a_0 + a_1\sigma + \cdots + a_5\sigma^5 \in \mathbb{Z}[G]$ is represented by the vector $[a_0, a_1, \ldots, a_5]$.

・ロン ・回 と ・ ヨ と ・ ヨ と

-2

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Let
$$\mathfrak{g}$$
 be a generator of $\operatorname{Cl}_k(3\infty_1\infty_2)$.
Let $\sigma := \sigma_{\mathfrak{g}}$. Thus $G = \langle \sigma \rangle$.
Let $\zeta_6 := \exp(2i\pi/6)$.

The character χ_a represented by [a] is the one defined by

$$\chi_a(\sigma) := \zeta_6^a.$$

An element $a_0 + a_1\sigma + \cdots + a_5\sigma^5 \in \mathbb{Z}[G]$ is represented by the vector $[a_0, a_1, \dots, a_5]$.

・ロン ・回 と ・ ヨ と ・ ヨ と

-

Statement of the conjecture	The example	
Current status of the conjecture	The strategy	
Checking the conjecture on an example	The verification	

Let
$$\mathfrak{g}$$
 be a generator of $\operatorname{Cl}_k(3\infty_1\infty_2)$.
Let $\sigma := \sigma_{\mathfrak{g}}$. Thus $G = \langle \sigma \rangle$.
Let $\zeta_6 := \exp(2i\pi/6)$.

The character χ_a represented by [a] is the one defined by

$$\chi_{a}(\sigma) := \zeta_{6}^{a}.$$

An element $a_0 + a_1\sigma + \cdots + a_5\sigma^5 \in \mathbb{Z}[G]$ is represented by the vector $[a_0, a_1, \ldots, a_5]$.

(4 同) (4 回) (4 回)

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Let \mathfrak{p} be a prime ideal of k, \mathfrak{P} a prime ideal of K such that

\mathfrak{P} is above \mathfrak{p} is above p.

Let $\theta \in K$ such that $K = \mathbb{Q}(\theta)$ and assume that

 $p \nmid (\mathbb{Z}_K : \mathbb{Z}[\theta]).$

Then the Frobenius of \mathfrak{p} is the unique element $\sigma \in G$ such that

 $\sigma(\theta) \equiv \theta^{\mathcal{N}\mathfrak{p}} \pmod{\mathfrak{P}}.$

소리가 소리가 소문가 소문가

-

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Let \mathfrak{p} be a prime ideal of k, \mathfrak{P} a prime ideal of K such that

 \mathfrak{P} is above \mathfrak{p} is above p.

Let $heta \in K$ such that $K = \mathbb{Q}(heta)$ and assume that

 $p \nmid (\mathbb{Z}_{K} : \mathbb{Z}[\theta]).$

Then the Frobenius of \mathfrak{p} is the unique element $\sigma \in G$ such that

 $\sigma(\theta) \equiv \theta^{\mathcal{N}\mathfrak{p}} \pmod{\mathfrak{P}}.$

-

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Let \mathfrak{p} be a prime ideal of k, \mathfrak{P} a prime ideal of K such that

 \mathfrak{P} is above \mathfrak{p} is above p.

Let $\theta \in K$ such that $K = \mathbb{Q}(\theta)$ and assume that

 $p \nmid (\mathbb{Z}_{K} : \mathbb{Z}[\theta]).$

Then the Frobenius of \mathfrak{p} is the unique element $\sigma \in G$ such that

$$\sigma(heta) \equiv heta^{\mathcal{N}\mathfrak{p}} \pmod{\mathfrak{P}}.$$

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Recall that $w_K = 6$ so the torsion group of K is generated by ζ_6 . Let $N \in \mathbb{Z}$ be such that

$$\sigma(\zeta_6) = \zeta_6^N.$$

Then an element $\alpha \in K$ is such that $K(\alpha^{1/6})/k$ is an abelian extension iff

$$\alpha^{N-\sigma} = \frac{\alpha^N}{\sigma(\alpha)}$$

is a 6-th power in K.

Statement of the conjecture	The example
Current status of the conjecture	The strategy
Checking the conjecture on an example	The verification

Recall that $w_{\mathcal{K}} = 6$ so the torsion group of \mathcal{K} is generated by ζ_6 . Let $\mathcal{N} \in \mathbb{Z}$ be such that

$$\sigma(\zeta_6) = \zeta_6^N.$$

Then an element $\alpha \in K$ is such that $K(\alpha^{1/6})/k$ is an abelian extension iff

$$\alpha^{N-\sigma} = \frac{\alpha^N}{\sigma(\alpha)}$$

is a 6-th power in K.