Artin L-functions

Charlotte Euvrard

Laboratoire de Mathématiques de Besançon

January 10, 2014

Definition

L/K Galois extension of number fields
$G = \mathrm{Gal}(L/K)$ its Galois group

Let (ρ, V) be a representation of G and χ the associated character.

Let $\mathfrak{p}$ be a prime ideal in the ring of integers $\mathcal{O}_K$ of K, we denote by :
$I_\mathfrak{p}$ the inertia group
$\varphi_\mathfrak{p}$ the Frobenius automorphism
$V^{I_\mathfrak{p}} = \{v \in V : \forall i \in I_\mathfrak{p}, \rho(i)(v) = v\}$

Definition

The Artin L-function is defined by :

$$L(s,\chi,L/K) = \prod_{\mathfrak{p}\subset O_K} \frac{1}{\det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\varphi_{\mathfrak{p}};\, V^{I_{\mathfrak{p}}})}, \qquad \mathrm{Re}(s) > 1,$$

where

$$\det(\mathrm{Id}-\mathrm{N}(\mathfrak{p})^{-s}\varphi_{\mathfrak{p}};\, V^{I_{\mathfrak{p}}}) = \begin{cases} \det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\rho(\varphi_{\mathfrak{p}})) & \text{if } \mathfrak{p} \text{ is unramified} \\ \det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\tilde{\rho}(\varphi_{\mathfrak{p}})) & \text{if } \mathfrak{p} \text{ is ramified and } V^{I_{\mathfrak{p}}} \neq \{0\} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\tilde{\rho}(\bar{\sigma}) = \frac{1}{|I_{\mathfrak{p}}|} \sum_{j\in I_{\mathfrak{p}}} \rho(j\sigma)$$

Definition

The Artin L-function is defined by :

$$L(s,\chi,L/K) = \prod_{\mathfrak{p}\subset O_K} \frac{1}{\det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\varphi_{\mathfrak{p}};\, V^{I_{\mathfrak{p}}})}, \qquad \mathrm{Re}(s) > 1,$$

where

$$\det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\varphi_{\mathfrak{p}};\, V^{I_{\mathfrak{p}}}) = \begin{cases} \det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\rho(\varphi_{\mathfrak{p}})) & \text{if } \mathfrak{p} \text{ is unramified} \\ \det(\mathrm{Id} - \mathrm{N}(\mathfrak{p})^{-s}\tilde{\rho}(\varphi_{\mathfrak{p}})) & \text{if } \mathfrak{p} \text{ is ramified and } V^{I_{\mathfrak{p}}} \neq \{0\} \\ 1 & \text{otherwise} \end{cases}$$

and

$$\tilde{\rho}(\bar{\sigma}) = \frac{1}{|I_{\mathfrak{p}}|} \sum_{j\in I_{\mathfrak{p}}} \rho(j\sigma)$$

In the particular case $K = \mathbb{Q}$: $L(s,\chi,L/\mathbb{Q}) = \prod_{p\in\mathbb{Z}} \frac{1}{\det(\mathrm{Id} - p^{-s}\varphi_p;\, V^{I_p})}$

Sum and Euler product

Proposition

Let d and d' be the degrees of the representations (ρ, V) and $(\tilde{\rho}, V^{I_p})$. Let $\alpha_{i,\rho}(p)$, $1 \leqslant i \leqslant d$ be the eigenvalues of $\tilde{\rho}(\varphi_p)$ with $\alpha_{i,\rho}(p) = 0$ if $d' < i \leqslant d$.

Then :

$$L(s,\chi,L/K) = \prod_{p \in \mathbb{Z}} \prod_{i=1}^{d} \left(1 - \alpha_{i,\rho}(p) p^{-s}\right)^{-1} = \sum_{n \geqslant 1} a_\chi(n) n^{-s}.$$

In particular, $|\alpha_{i,\rho}(p)| = 1$ for $1 \leqslant i \leqslant d'$.

Properties

1. *We have :*

$$L(s, 1_G, L/\mathbb{Q}) = \zeta(s),$$

   where ζ is the Riemann zeta function.

2. *For χ_1, χ_2 two characters of G,*

$$L(s, \chi_1 + \chi_2, L/\mathbb{Q}) = L(s, \chi_1, L/\mathbb{Q}) L(s, \chi_2, L/\mathbb{Q}).$$

3. *We can write :*

$$\zeta_L(s) = \zeta(s) \prod_{\chi \neq 1} L(s, \chi, L/\mathbb{Q})^{\chi(1)}$$

   where ζ_L is the Dedekind zeta function of L.

Examples

Functional equation

Definition

Define :

$$\gamma_\chi(s) = \left(\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\right)^{\dim V^+}\left(\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)\right)^{\dim V^-}$$

for every w of L above the infinite place of $\mathbb{Q}$: the generator σ_w of $G(w) = \{g \in G \mid \rho(g)(w) = w\}$ induces an eigenspace decomposition $V = V^+ \oplus V^-$ where V^+ and V^- correspond to the eigenvalues $+1$ and -1 of $\rho(\sigma_w)$. Actually,
$\dim V^+ = \dim V^{<1,\sigma_w>} = \frac{1}{2}(d + \chi(\sigma_w))$ *and so*
$\dim V^- = \frac{1}{2}(d - \chi(\sigma_w))$.

Artin conductor

Definition

Let G_i be the i-th ramification group of $\mathfrak{p}$ above $p \in \mathbb{Z}$.
Define :

$$f_p(\chi) = \sum_{i=0}^{+\infty} \frac{|G_i|}{|G_0|} \operatorname{codim} V^{G_i}.$$

Definition (Artin conductor)

The Artin conductor is :

$$f(\chi) = \prod_{p \in \mathbb{Z}} p^{f_p(\chi)}.$$

Functional equation

Theorem

Functional equation Let $\Lambda(s,\chi) = f(\chi)^{\frac{s}{2}}\gamma_\chi(s)L(s,\chi,L/\mathbb{Q})$ *be the completed Artin L-function.*
Then $\Lambda(s,\chi)$ *admits a meromorphic continuation to* $\mathbb{C}$*, analytic except for poles at* $s = 0$ *and* $s = 1$ *and satisfies :*

$$\Lambda(1-s,\chi) = W(\chi)\Lambda(s,\overline{\chi}),$$

with a constant $W(\chi)$ *of absolute value 1.*

Functional equation

Theorem

Functional equation Let $\Lambda(s,\chi) = f(\chi)^{\frac{s}{2}}\gamma_\chi(s)L(s,\chi,L/\mathbb{Q})$ *be the completed Artin L-function.*
Then $\Lambda(s,\chi)$ *admits a meromorphic continuation to* $\mathbb{C}$*, analytic except for poles at* $s=0$ *and* $s=1$ *and satisfies :*

$$\Lambda(1-s,\chi) = W(\chi)\Lambda(s,\overline{\chi}),$$

with a constant $W(\chi)$ *of absolute value 1.*

Property

We can write :

$$\gamma_\chi(s) = \pi^{-\frac{ds}{2}}\Gamma\left(\frac{s}{2}\right)^{\dim V^+}\Gamma\left(\frac{s+1}{2}\right)^{\dim V^-}.$$

Conjectures

The trivial zeroes of $L(s,\chi)$ occur at negative integers.

The trivial zeroes of $L(s,\chi)$ occur at negative integers.
The nontrivial zeroes of $L(s,\chi)$ lie in the open strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$.

The trivial zeroes of $L(s,\chi)$ occur at negative integers.
The nontrivial zeroes of $L(s,\chi)$ lie in the open strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$.

Grand Riemann Hypothesis

The nontrivial zeros of Artin L-functions lie on the critical line $1/2 + \mathrm{i}t$.

Conjectures

The trivial zeroes of $L(s,\chi)$ occur at negative integers.
The nontrivial zeroes of $L(s,\chi)$ lie in the open strip $\{s \in \mathbb{C} : 0 < \mathrm{Re}(s) < 1\}$.

Grand Riemann Hypothesis

The nontrivial zeros of Artin L-functions lie on the critical line $1/2 + it$.

Artin conjecture

For each nontrivial irreducible character, $L(s,\chi)$ is entire.

Examples