Elliptic curves

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1 Riemman ζ function

Exercise 1.

1. Give the list of the 100 first coefficients of the Riemman ζ function.
2. Check that $\zeta(2) = \frac{\pi^2}{6}$.
3. Give the list of zeros z on the critical line with $0 \leqslant \Im(z) \leqslant 30$.
4. Use `zeta` to compute some values of the ζ-function. Compare with `lfun`.
5. Create a function `gammafactor(s)` which returns the value of $\gamma_A(s) = \prod_i \Gamma_{\mathbb{R}}(s + a_i)$ (see `gamma`).
6. Check the functional equation on the Λ .

Exercise 2.
Numerically check Riemann's hypothesis (see `ploth`).

2 Dirichlet L-functions

Exercise 3.
Choose your favourite modulus m.

1. Give the structure of $(\mathbb{Z}/m\mathbb{Z})^*$.
2. Find a primitive character χ of G (see `zncharconductor`).
3. Create $L(s, \chi)$.
4. Check the following :
   - If χ is even, then $L(s, \chi)$ has simple zeros at $s = 0, -2, -4, \ldots$.
   - If χ is odd, then $L(s, \chi)$ has simple zeros at $s = -1, -3, -5 \ldots$.

3 L-function of an elliptic curve over a number field

Exercise 4. We define the elliptic curve $E : y^2 + xy + \phi x = x^3 + (\phi + 1)x^2 + x$ over the field $K = \mathbb{Q}(\sqrt{5})$ where $\phi = \frac{1+\sqrt{5}}{2}$.

1. Initialize the number field K, then the elliptic curve E.
2. Compute its j-invariant, discriminant, conductor and its torsion.
3. Check the BSD conjecture for E (see `elltalmagama` and `E.omega`).
4. Create $L(E, s)$.
5. Check the functional equation : $\Lambda(E, s) = \varepsilon N^{1-s}\Lambda(E, 2 - s)$, where $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^s\Gamma(s)$ and $\Lambda(E, s) := \Gamma_{\mathbb{C}}(s)L(E, s)$.