Miller Inversion is Easy for the Reduced Tate Pairing of Embedding Degree Greater than one

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Dedicated to Professor Nobushige Kurokawa
on his 73rd birthday

Abstract

Let q be a power of an odd prime p. Denote the finite field of q elements by $\mathbf{F}_q$. We present a polynomial time algorithm for Miller inversion in the reduced Tate pairing inversion on an elliptic curve E over $\mathbf{F}_q$ with an embedding degree $k>1$. Assume that we precomputed a generator of the 2-Sylow subgroup of $\mathbf{F}_q^\times$ which depends only on q. In case of even k, our algorithm is runs deterministically with $O((k\log q)^3)$ bit operations. In case of odd k, our algorithm is runs probabilistically with $O(k^6(\log q)^3)$ bit operations in average.

1. Introduction

Difficulty of pairing inversion is a fundamental assumption in pairing based cryptography. Galbraith, Hess and Vercauteren[9, end of Chap. 3] gives explicit description how pairing inversions break Hess' IBS. Duc and Jetchev[6, Sect. 5.2] gives explicit description how pairing inversions break Boneh-Franklin's IBE and Joux's tripartite key agreement protocol. For the Kate, Zaverucha and Goldberg polynomial commitment protocol[13], one pairing inversion with some group operations in the domain or the range of the pairing produces a false witness for their verify evaluation algorithm. More interestingly, Verheul[18] proved that the computational Diffie-Hellman problem is reduced to pairing inversion. The result is extended to asymmetric pairings by Karabina, Knapp and Menezes[12].

Galbraith, Hess and Vercauteren[9] proposed a two step pairing inversion framework. The first step is called final exponentiation inversion (FEI), while the second step is called Miller inversion (MI). In general, both steps have been considered to be difficult. However, [9, Sect. 6] proposes a family of pairing friendly elliptic curves whose MI are easy. Akagi and Nogami[1] proved that MI are easy for Barreto-Naehrig curves[3] of embedding degree 12, Brezing-Weng[5] curves of embedding degree 8 and Freeman curve[7] of embedding degree 10. Assuming the Bateman and Horn conjecture[4] which is plausible but unproved, we see that these families consists of infinitely many elliptic curves.

The purpose of this short note is to present a feasible algorithm for MI for the reduced Tate pairing for any embedding degree greater than 1. More specifically, let q be a power of an odd prime p. Denote the finite field of q elements by $\mathbf{F}_q$. Let E be an elliptic curve defined over $\mathbf{F}_q$ by the Wierestrass model. Let l be a prime di-

visor of $^{\#}E(\mathbf{F}_q)$ different from p. Let k be the embedding degree of l for $E/\mathbf{F}_q$. That is, k is the minimal positive integer satisfying $l \mid q^k - 1$. We assume $k > 1$. Let $A \in E(\mathbf{F}_{q^k})$ be of order l satisfying $\sigma_q(A) = qA$ where σ_q is the qth power Frobenius endomorphism. Let $h_{l,A}$ is the l-th Miller function for A. For $z \in \mathbf{F}_{q^k}^{\times}$, we give an algorithm to find $Q \in E(\mathbf{F}_q)$ satisfying $h_{l,A}(Q) = z$ if such Q exists. If such Q does not exist, our algorithm reports the nonexistence. Note that $h_{l,A}(Q)^{(q^k-1)/l}$ is the reduced Tate pairing of A and Q. Assume that we precomputed a generator of the 2-Sylow subgroup of $\mathbf{F}_q^{\times}$ which depends only on q.[1] Then, time complexity of our algorithm is $O((k \log q)^3)$ bit operations for even k and $O(k^6 (\log q)^3)$ bit operations in average for odd $k > 1$.

We note that supersingular curves gives infinitely many instances for our algorithm even if we fix a field characteristic p. Let G be a prime order subgroup of the unit group of the algebraic closure of $\mathbf{F}_p$ and let k be the smallest positive integer satisfying $G \subset \mathbf{F}_{p^k}$. Assume $3 \mid k$ and $p \equiv 5 \bmod 6$. Then the computational Diffie-Hellman problem is reduced to the reduced Tate pairing inversion problem of some supersingular curves E of embedding degree 3 over a finite extension of $\mathbf{F}_p$, provided if such E exists. Sufficient conditions for existence of such E and its construction are described in Section 2. Here we observe that a property of a single embedding degree for supersingular curves is applicable to infinitely many k. A similar property holds for the case $2 \mid k$ and $p \equiv 3 \bmod 4$.

The rest of the paper is organized as follows: Section 2 summarizes some properties of supersingular curves. Section 3 describes MI for supersingular curves with embedding degree three. In this case we utilize the fact that the Frobenius endomorphism acts trivially on the Y-coordinate of some points, which makes our algorithm simple. In Section 4, we present our algorithm for even embedding degrees, which is deterministic. In Section 5, we present our algorithm for odd embedding degrees, which is probabilistic. In both Sections 4 and 5, a property of the Ate pairing due to Granger et al.[10, Theorem 2] plays an essential role.

If we exclude side-channel attacks (and use of quantum computers), FEI seems to be a very hard problem. See Vercauteren[17]. If FEI is actually a hard problem, our result has probably no impact to real world cryptography. However, Lashermes, Fournier and Goubin[14] gives a fault attack method for FEI. Although their method is intended for ordinary curves, it is also applicable to supersingular curves. Indeed, the method described in Section 4.2 of [14] is sufficient for the embedding degree two case. Thus, if one has concerns about fault attacks, final exponentiation must be so implemented that it is immune to such attacks.

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[1] This is used in square root computations to recover the Y coordinate of Q form the X coordinate of Q.

Notation.

Throughout this note, an elliptic curve E is given by the Weierstrass form. The X and Y coordinate functions are denoted by ξ and η, respectively. We use $\tau := -\xi/\eta$ as a local parameter at the point $\mathcal{O}$ at the infinity of E. For a rational function f on E, we denote by $\mathrm{ord}_{\mathcal{O}} f$ the order of zero of f at $\mathcal{O}$ (negative if f has a pole at $\mathcal{O}$). We also denote by $\mathrm{lc}(f)$ the leading coefficient of Laurent series expansion of f at $\mathcal{O}$ w.r.t. τ, i.e.,

$$f = \mathrm{lc}(f)\tau^m + O(\tau^{m+1})$$

where $m = \mathrm{ord}_{\mathcal{O}} f$. A rational function f is said to be normalized if $\mathrm{lc}(f) = 1$. For $\varrho \in \mathrm{End}(E)$, we define $\mathrm{lc}(\varrho) := \mathrm{lc}(\tau \circ \varrho)$. Note that $\mathrm{lc}(f)$ depends on a choice of a local parameter at $\mathcal{O}$, whereas $\mathrm{lc}(\varrho)$ does not. For an object over a field of characteristic p, we write p^n-th power Frobenius as φ_{p^n}. Finally, for $P \in E$ and $n \in \mathbf{N}$, we denote by $h_{n,P}$ the normalized n-th Miller function for P, i.e., the normalized rational function satisfying $\mathrm{div} h_{n,P} = n[P] - [nP] - (n-1)[\mathcal{O}]$.

2. Supersingular Curves

We construct some supersingular elliptic curves used in reducing certain computational Diffie-Hellman problems to pairing inversion problems. Their construction is well known or easily derived from well known results. Let p be a prime. For an integer n which is co-prime to p, we denote by μ_n the group of n-th roots of unity in $\mathbf{F}_p^a$ where $\mathbf{F}_p^a$ is an algebraic closure of $\mathbf{F}_p$. Let M be a finite extension of $\mathbf{F}_p$ and put $K := M(\mu_n)$. Let E/M be an elliptic curve. For $P \in E(K)[n]$ and $Q \in E(K)$, the n-th reduced Tate pairing is defined by

$$\langle P, Q \rangle_n := h_{n,P}(Q)^{\#(K^\times)/n}.$$

In our application, $E[n] \subset E(K)$ and it induces a bilinear pairing $E[n] \times E[n] \to \mu_n$.

Lemma 2.1. *Assume $p \equiv 5 \bmod 6$. Let $l \geq 5$ be a prime and let k be the smallest positive integer satisfying $\mu_l \subset \mathbf{F}_{p^k}^\times$. Assume that k is divisible by 3.*[1] *Put*

$$q' := \begin{cases} p^{k/3} & (k/3 \text{ is odd}), \\ p^{k/6} & (k/3 \text{ is even}), \end{cases}$$

and $q := q'^2$. Then there exists a supersingular curve $E/\mathbf{F}_q$ satisfying the followings.

(1) $l \mid \#E(\mathbf{F}_q)$.

(2) $E[l] \subset E(\mathbf{F}_{q^3})$.

(3) $\mu_l \subset \mathbf{F}_{q^3}^\times$.

(4) $j(E) = 0$.

Proof. First we prove existence of E satisfying (1)–(3) in case that $k/3$ is odd. The assertion (3) is obvious since $q^3 = p^{2k}$. Put $t := -q'$ and $N := q - t + 1$. Since l is prime, either $l \mid q' - 1$ or $l \mid q + q' + 1 = N$. By the minimality of k, we see $l \mid N$. By Waterhouse[20, Theorem 4.1], there exists a supersingular curve $E/\mathbf{F}_q$ whose trace of

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[1] Note that, for fixed p, there exist infinitely many l satisfying the conditions. The same remark applies to Lemma 2.2.

the q-th power Frobenius φ_q is t. Hence $^{\#}E(\mathbf{F}_q)=N$, which proves (1). Suppose $l \mid q-1$. Then, we have

$$l \mid \gcd(q-1, q+q'+1) = \gcd(q-1, q'+2) = \gcd(q-1-(q'^2-4), q'+2) = \gcd(3, q'+2) = 1,$$

a contradiction. By Balasubramanian and Koblitz[2, Theorem 1], we see (2) holds.

Next, we prove existence of E satisfying (1)-(3) in case that $k/3$ is even. Again, the assertion (3) is obvious since $q^3=p^k$. Put $t:=q'$ and $N:=q-t+1$. We see l divides one of $q'-1$, $q'+1$, $q+t+1$, N. However $l \mid q+t+1$ implies $\mu_l \subset \mathbf{F}_{p^{k/2}}^{\times}$ which contradicts the minimality of k. Similarly, we see $l \nmid q'\pm 1$. Thus $l \mid N$. The existence of E and the assertion (1) follow from the same argument as above. The assertion (2) holds because

$$\gcd(q-1, q-q'+1) = \gcd(q-1, -q'+2) = \gcd(q-1-(q'^2-4), q'-2) \mid 3$$

implies $l \nmid q-1$.

Finally, we prove (4). Let t and N be as above. Since $[q']$ is purely inseparable, there exists $\omega \in \mathrm{Aut}(E)$ satisfying $[q']=\omega\varphi_q$. Then $\varphi_q^2 - t\varphi_q + q = 0$ in $\mathrm{End}(E)$ implies that

$$\omega^2 - \mathrm{sgn}(t)\omega + 1 = 0. \tag{2.1}$$

This shows that $\mathrm{sgn}(t)\mathrm{lc}(\omega)$ $(\in \mathbf{F}_q^{\times})$ is a primitive 6th root of unity. Therefore $^{\#}\mathrm{Aut}(E)=6$, which proves $j(E)=0$ (see e.g. Silverman[16, Sect. III.10]). □

Lemma 2.2. *Assume $p \equiv 3 \bmod 4$. Let l be an odd prime and let k be the smallest positive integer satisfying $\mu_l \subset \mathbf{F}_{p^k}^{\times}$. Assume that k is divisible by 2. Put $q := p^{k/2}$. Then there exists a supersingular curve $E/\mathbf{F}_q$ satisfying the followings.*

(1) $l \mid {}^{\#}E(\mathbf{F}_q)$.

(2) $E[l] \subset E(\mathbf{F}_{q^2})$.

(3) $\mu_l \subset \mathbf{F}_{q^2}^{\times}$.

(4) $j(E)=1728$.

Proof. Existence of E satisfying (1)-(3) are proved by a similar (in fact easier) method to the proof of Lemma 2.1. We prove (4). In case that $k/2$ is odd, any elliptic curve E defined over $\mathbf{F}_q$ satisfying $j(E)=1728$ is supersingular. We *choose* such a curve as E. In case that $k/2$ is even, the unique automorphism ω satisfying $[\sqrt{q}]=\omega\varphi_q$ satisfies $\omega^2+1=0$ in $\mathrm{End}(E)$. Hence $^{\#}\mathrm{Aut}(E)=4$ and $j(E)=1728$. □

Once we have proved $j(E)=0$ or $j(E)=1728$, we can easily construct an explicit Weierstrass model for E and its distortion map from Galbraith[8, Table IX.1] with some modifications. Just for completeness, we list them. For a field K and $n \in \mathbf{N}$, we put $K^{\neg n} := K - \{x^n : x \in K\}$.

k	p	Weierstrass model	distortion map
$3 \mid k$	$5 \bmod 6$	$Y^2=X^3+c,\ c \in \mathbf{F}_q^{\neg 3}$.	$(\gamma\zeta^p,\ c^{-(p-1)/2}\eta^p),\ \gamma \in \mathbf{F}_{q^3}^{\times}$ s.t. $\gamma^3=c^{-(p-1)}$.
$k \equiv 2 \bmod 4$	$3 \bmod 4$	$Y^2=X^3-cX,\ c \in \mathbf{F}_q^{\times}$.	$(-\zeta,\ \gamma\eta),\ \gamma \in \mathbf{F}_{q^2}^{\times}$ s.t. $\gamma^2=-1$.
$4 \mid k$	$3 \bmod 4$	$Y^2=X^3-cX,\ c \in \mathbf{F}_q^{\neg 2}$.	$(c^{-(p-1)/2}\zeta^p,\ \gamma\eta^p),\ \gamma \in \mathbf{F}_{q^2}^{\times}$ s.t. $\gamma^2=c^{-3(p-1)/2}$.

Note that $\gamma \notin \mathbf{F}_q^{\times}$ in the all cases. Note also that powering is not a q-th power but a

p-th power in the first and the third case. In the first case, the followings are equivalent: $\mathrm{Tr}(\varphi_q)=q'$, $3 \mid {}^{\#}E(\mathbf{F}_q)$, $E[3](\mathbf{F}_q)\neq\{\mathcal{O}\}$, c is square in $\mathbf{F}_q$ (consider the third division polynomial).

The above table and Karabina, Knapp and Menezes[12, Theorem 3] summarizes to the following statement.

Proposition 2.3. *Let p, G, k and E be as above. Assume that $2 \mid k$ and $p \equiv 3 \bmod 4$ or that $3 \mid k$ and $p \equiv 5 \bmod 6$. Then, the computational Diffie-Hellman problem on G is reduced to the reduced Tate pairing inversion on E in probabilistic polynomial time with respect to ${}^{\#}G$.*

3. The case of Embedding Degree Three

In this section, we consider the Miller inversion for the case that embedding degree is three. Let p be a prime satisfying $p \equiv 5 \bmod 6$ and let q be an even power of p. We put $q' := \sqrt{q} \in \mathbf{N}$. Let t be either q' or $-q'$. Let $E/\mathbf{F}_q$ be a supersingular elliptic curve of $\mathrm{Tr}(\varphi_q)=t$, given by the *short* Weierstrass form. Define $\omega \in \mathrm{Aut}(E)$ by $[q']=\omega\varphi_q$, as in Lemma 2.1. Note $j(E)=0$ and

$$\omega = (\mathrm{lc}(\omega)^{-2}\xi, \mathrm{lc}(\omega)^{-3}\eta) \tag{3.1}$$

(see Silverman[16, Sect. III.10] for example). Put $N := q-t+1$ and $r := q^3$. By Schoof[15, Lemma 4.8], $E(\mathbf{F}_q) \cong \mathbf{Z}/N\mathbf{Z}$. The embedding degree for $E[N]$ is 3. However we note that in case of $t=-q'$, the minimal embedding field in the sense of Hitt[11] is not $\mathbf{F}_r$ but $\mathbf{F}_{q^3}$.

Let l be an divisor of N, which is not necessarily a prime in this section. In case of $t \equiv 2 \bmod 3$, we further assume that l is not divisible by 3. Put $G_1 := E[l] \cap E(\mathbf{F}_q)$ and $G_0 := \{ P \in E[l] : \varphi_q P = qP \}$. Then $G_1 \cap G_0 = \{\mathcal{O}\}$ and $E[l] = G_1 \oplus G_0$ since $\gcd(q-1, l) \mid \gcd(3, t-2)$ (cf. proof of Lemma 2.1). In particular, G_0 is also a cyclic group of order l. For $A \in G_0$, observe $[q']A = \omega\varphi_q A = q\omega A$, hence $\omega^{-1}A = [q']A$ and

$$\omega^{-2}A = qA = \varphi_q A. \tag{3.2}$$

Observe that $\mathrm{lc}(\omega)^{-2}$ is a primitive cubic root of the unity (cf. (2.1) and below). Then (3.1) and (3.2) imply

$$\eta \circ \omega^{-2} = \eta, \tag{3.3}$$

$$\eta(A) \in \mathbf{F}_q \text{ for } A \in G_0 - \{\mathcal{O}\} \tag{3.4}$$

and

$$\xi(\omega^{-2}A) \neq \xi(A) \text{ for } A \in G_0 - \{\mathcal{O}\}. \tag{3.5}$$

(Otherwise, $(1-\mathrm{lc}(\omega)^4)\xi(A)=0$. Hence $\xi(A)=0$ and $A \in E(\mathbf{F}_q)$, which contradicts to $G_0 \cap G_1 = \{\mathcal{O}\}$). Let ζ be a primitive 3rd root of the unity. Now we state our algorithm.

Algorithm 3.1.
Input: $v \in \mathbf{F}_r$, $A \in G_0 - \{\mathcal{O}\}$. // Note that A may not be a generator.
Output: $Q \in G_1 - \{\mathcal{O}\}$ satisfying $h_{N,A}(Q) = v$ if such Q exists. Otherwise, **nil**.

Procedure:

1: $u := v^{(1+q+q^2)(2+t)/3}$;
2: if $u \notin \mathbf{F}_q$ then return **nil** ;
3: $y_i := \eta(A) - \zeta^i u$ for $i = 1, 2, 3$.
4: Build a set $L_i := \{Q \in E(\mathbf{F}_q) : \eta(Q) = y_i\}$ for $i = 1, 2, 3$. // Note $0 \le {}^{\#}L_i \le 3$.
5: for each $Q \in L_1 \cup L_2 \cup L_3$
6: if $lQ = \mathcal{O}$ and $h_{N,A}(Q) = v$ then return Q ;
7: return **nil** ;

Before we evaluate computational complexity of our algorithm, we clarify assumptions on time complexities for operations on elements of $\mathbf{F}_q$ or $\mathbf{F}_r$. We assume that $\mathbf{F}_q$ and $\mathbf{F}_r$ are so realized that one arithmetic operation in $\mathbf{F}_q$ or $\mathbf{F}_r$ amounts to $O((\log q)^2)$ bit operations. We also assume that a generator g of 3-Sylow subgroup of $\mathbf{F}_q^\times$ is precomputed. This is achieved by a probabilistic algorithm which needs $O((\log q)^3)$ bit operations in average. Using g, we can deterministically compute a cubic root of a cubic element of $\mathbf{F}_q^\times$ with $O((\log q)^3)$ bit operations.

Theorem 3.2. *Algorithm 3.1 returns a correct result with $O((\log q)^3)$ bit operations.*

Proof. First, we prove correctness. Suppose there exists $Q \in G_1 - \{\mathcal{O}\}$ satisfying $h_{N,A}(Q) = v$. By the definition of the Miller function,

$$\begin{aligned} \operatorname{div} h_{N,A} &= N([A] - [\mathcal{O}]), \\ \operatorname{div} h_{N,\omega^{-2}A} &= N([\omega^{-2}A] - [\mathcal{O}]), \\ \operatorname{div} h_{N,\omega^{-4}A} &= N([\omega^{-4}A] - [\mathcal{O}]). \end{aligned} \tag{3.6}$$

Observe that $A + \omega^{-2}A + \omega^{-4}A = \mathcal{O}$. For A, $B \in E$, let, as usual, $L_{A,B}$ be the normalized rational function on E whose divisor is $[A]+[B]+[-(A+B)]-3[\mathcal{O}]$. Summing up both sides of (3.6), we obtain

$$\operatorname{div}(h_{N,A} h_{N,\omega^{-2}A} h_{N,\omega^{-4}A}) = N([A] + [\omega^{-2}A] + [\omega^{-4}A] - 3[\mathcal{O}]) = N \operatorname{div} L_{A,\omega^{-2}A}.$$

Since both functions $h_{N,A} h_{N,\omega^{-2}A} h_{N,\omega^{-4}A}$ and $L_{A,\omega^{-2}A}$ are normalized,

$$h_{N,A} h_{N,\omega^{-2}A} h_{N,\omega^{-4}A} = L_{A,\omega^{-2}A}^N.$$

By (3.5),

$$L_{A,\omega^{-2}A} = -\eta + \frac{\eta(\omega^{-2}A) - \eta(A)}{\xi(\omega^{-2}A) - \xi(A)}(\xi - \xi(A)) + \eta(A).$$

The term containing $\xi(A)$ vanishes by (3.3). Therefore

$$h_{N,A} h_{N,\omega^{-2}A} h_{N,\omega^{-4}A} = (-\eta + y)^N$$

where $y := \eta(A) = \eta(\omega^{-2}A) = \eta(\omega^{-4}A)$. Since E is defined over $\mathbf{F}_q$,

$$\varphi_q(h_{N,A}(Q)) = h_{N,\varphi_q A}(\varphi_q Q) = h_{N,\omega^{-2}A}(Q)$$

while by definition $\varphi_q h_{N,A}(Q) = h_{N,A}(Q)^q$. Taking (3.4) and $\eta(Q) \in \mathbf{F}_q$ into consideration, we obtain

$$\begin{aligned} h_{N,A}(Q)^{1+q+q^2} &= (\eta(A) - \eta(Q))^{q-t+1} = (\eta(A) - \eta(Q))^{2-t} \\ v^{(1+q+q^2)(2+t)} &= (\eta(A) - \eta(Q))^{(2-t)(2+t)} = (\eta(A) - \eta(Q))^{4-q} = (\eta(A) - \eta(Q))^3. \end{aligned}$$

Therefore $\eta(Q)$ is either y_1, y_2 or y_3. If $\eta(Q) = y_i$ then $Q \in L_i$ by the definition of L_i. Let R be any point in $L_1 \cup L_2 \cup L_3$. A priori $R \in E(\mathbf{F}_q)$. The tests in Step 6 ensures

that the algorithm terminates with an output R whenever $R \in G_1 - \{\mathcal{O}\}$ and $h_{N,A}(R) = v$. (Note that R may be different from Q.) This also implies that the algorithm reaches Step 7 only if there is no element Q in G_1 satisfying $h_{q+1,A}(Q) = v$.

Next, we evaluate computational complexity of Algorithm 3.1. Since $\frac{1+q+q^2}{3} \in \mathbf{N}$, Step 1 needs $O(\log q)$ multiplications in $\mathbf{F}_r$. For each i, we obtain L_i with $O(1)$ arithmetic operations and one cubic root computation in $\mathbf{F}_q$ (not in $\mathbf{F}_r$, which is ensured by Step 2). At Step 6, we have a point $Q \in E(\mathbf{F}_q)$. Since $G_0 \cap E(\mathbf{F}_q) = \{\mathcal{O}\}$ by the condition on l, no division by zero occurs during evaluation of $h_{N,A}(Q)$ by the Miller algorithm. Hence we obtain the value of $h_{N,A}(Q)$ with $O(\log q)$ arithmetic operations over $\mathbf{F}_r$. Thus the algorithm terminates with $O(\log q)$ arithmetic operations over $\mathbf{F}_r$ or $\mathbf{F}_q$ and at most nine cubic root computations in $\mathbf{F}_q$. By our assumptions, they amount to $O((\log q)^3)$ bit operations. □

Example 3.3. We consider the case $p := 11$, $t := 11$, $q := p^2$, $N := q - t + 1 = 111$ and $l := 37$. Let θ be the class of T in $\mathbf{F}_p[T]/\langle T^6 + T + 2 \rangle$ and put $i := 5\theta^5 + 9\theta^4 + 8\theta^3 + 7\theta^2 + \theta + 6$. We see $i^2 = -1$. So, we use $\mathbf{F}_p(\theta)$ and its subfield $\mathbf{F}_p(i)$ as $\mathbf{F}_{q^3}$ and $\mathbf{F}_q$, respectively. One of the primitive third roots of unity is $\zeta := 8i + 5$. Consider $E: Y^2 = X^3 + 8i + 4/\mathbf{F}_q$. We see $^{\#}E(\mathbf{F}_q) = N$. Put $A := (8\theta^5 + \theta^4 + 4\theta^3 + 8\theta^2 + 6\theta + 3,\ 7i) \in G_0$ and $v := \theta^5 - \theta^4 - 2\theta^2 - \theta - 1$. Then $u := v^{63973} = 6 + 6i \in \mathbf{F}_q$ and we obtain $y_1 := 6i + 7$, $y_2 := 3i - 1$, $y_3 := i + 5$. Then $L_1 = \emptyset$, $L_2 = \emptyset$, and $L_3 = \{(1+i, y_3),\ (2i+8, y_3),\ (8i+2, y_3)\}$. The Miller algorithm gives

$$\begin{aligned} h_{N,A}(1+i, y_3) &= 2\theta^4 + \theta^3 + 8\theta^2 + 6\theta, \\ h_{N,A}(2i+8, y_3) &= 10\theta^5 + 10\theta^4 + 10\theta^3 + 5\theta^2 + 6\theta + 1, \\ h_{N,A}(8i+2, y_3) &= \theta^5 - \theta^4 - 2\theta^2 - \theta - 1. \end{aligned}$$

Therefore we obtain the desired answer $Q := (8i+2, i+5)$.

4. The case of Even Embedding Degrees

Let p be an odd prime and let q be a power of p. Let E be an elliptic curve defined by the Wierstrass model over $\mathbf{F}_q$. Let l be an odd divisor $^{\#}E(\mathbf{F}_q)$ which is prime to p. We denote the embedding degree of l for E by k. That is, k is the minimal positive integer satisfying $l \mid q^k - 1$. Throughout this section, we assume that k is even. We further assume that $l \mid q^{k/2} + 1$. This condition is automatically satisfied in ether the case that l is a prime or the case that E is supersingular.[1] Put $r := q^k$ and $s := q^{k/2}$ for simplicity. Put $G_1 := E[l] \cap E(\mathbf{F}_q)$ and $G_0 := \{P \in E[l] : \varphi_s(P) = sP\}$. We have $G_0 \cap G_1 = \{\mathcal{O}\}$ since l is odd. In stead of (3.2), we have

$$\varphi_s A = -A$$

which implies $\xi(A) \in \mathbf{F}_s$ for $A \in G_0$ in the even embedding degree case. We assume that we precomputed a generator of 2-Sylow subgroup of $\mathbf{F}_q^{\times}$ which is used in square root computations. Our algorithm for an even embedding degree is as follows:

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[1] By Schoof[15, Lemma 4.8], $E(\mathbf{F}_r) = E[q+1]$ and $E(\mathbf{F}_q)$ is isomorphic to either $\mathbf{Z}/(q+1)\mathbf{Z}$ or $\mathbf{Z}/\left(\frac{q+1}{2}\right)\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$. Hence $l \mid q+1$ if E is supersingular.

Algorithm 4.1.
Input: $d \in \mathbf{N}$ satisfying $l \mid d$ and $d \mid (s+1)$,
$v \in \mathbf{F}_r$,
$A \in G_0 - \{\mathcal{O}\}$. // Note that A may not be a generator (if l is composite).
Output: $Q \in G_1 - \{\mathcal{O}\}$ satisfying $h_{d,A}(Q) = v$ if such Q exists. Otherwise, **nil**.
Procedure:
1: $u := (v^{(s+1)/d})^{(s+1)/2}$;
2: if $u \notin \mathbf{F}_s$ then return **nil** ;
3: $x_1 := \xi(A) + u$; $x_2 := \xi(A) - u$;
4: Build a set $L_i := \{ Q \in E(\mathbf{F}_q) : \xi(Q) = x_i \}$ for $i = 1, 2$. // Note $0 \le {}^{\#}L_i \le 2$.
5: for each $Q \in L_1 \cup L_2$
6: if $lQ = \mathcal{O}$ and $h_{d,A}(Q) = v$ then return Q ;
7: return **nil** ;

Theorem 4.2. *Algorithm 4.1 returns a correct result with* $O((k \log q)^3)$ *bit operations.*

Proof. First, we prove correctness. Suppose there exists $Q \in G_1 - \{\mathcal{O}\}$ satisfying $h_{d,A}(Q) = v$. Recall that E is defined by the Weierstrass model. Since $l \mid d$, we have $h_{s+1,A} = h_{d,A}^{(s+1)/d}$. Since $A \in G_0 \subset E[s+1]$, we have

$$h_{s+1,A} = (\xi - \xi(A)) h_{s,A}. \tag{4.1}$$

Now key observation of our algorithm is $h_{s,A}(Q) \in \mu_l \subset \mu_{s+1}$ by Granger et al.[10, Theorem 2]. Thus evaluation of (4.1) at Q followed by $s+1$ powering yields

$$(v^{(s+1)/d})^{s+1} = (\xi(Q) - \xi(A))^{s+1}. \tag{4.2}$$

That is, we do not need the value $h_{s,A}(Q)$ at all. Since $Q \in E(\mathbf{F}_q) - \{\mathcal{O}\}$, we have $\xi(Q) \in \mathbf{F}_q$. On the other hand $A \in G_0 - \{\mathcal{O}\}$ implies $\xi(A) = \varphi_s(\xi(A))$. Thus $\xi(A) \in \mathbf{F}_s$. Therefore $\xi(Q) - \xi(A) \in \mathbf{F}_s$ and $(\xi(Q) - \xi(A))^{s+1} = (\xi(Q) - \xi(A))^2$. Substituting the right side of (4.2), we obtain

$$v^{(s+1)^2/d} = (\xi(Q) - \xi(A))^2. \tag{4.3}$$

Recall that q is odd. Hence

$$\xi(Q) - \xi(A) = \pm v^{(s+1)^2/2d} = \pm u.$$

Therefore $\xi(Q)$ is either x_1 or x_2. Since l is odd, $G_0 \cap E(\mathbf{F}_s) = \{\mathcal{O}\}$. The rest of proof of correctness and a proof for computational complexity are similar to the proof of Theorem 3.2. □

Remark 4.3. In case that q is a power of 2, the algorithm and its implementation are in fact easier because (4.3) yields a unique candidate of $\xi(Q)$. However in cryptographic point of view, this case is irrelevant.

Example 4.4. Consider $E : Y^2 = X^3 - 13X - 7$ over $\mathbf{F}_{139}$ and take $l := 35$. So, $q = p = 139$ and $k = 2$. Put $d := 140$. Let θ be the class of T in $\mathbf{F}_{139}[T]/\langle T^2 + 4 \rangle$. Then $\mathbf{F}_{139^2} = \mathbf{F}_{139}(\theta)$. Put $A := (67, 38\theta)$ and $v := 25\theta + 109$. Note that $\langle A \rangle = G_0$ and that v^{138} is a primitive 35-th root of unity. Then $u := v^{70} = 131$ and we obtain $x_1 := 59$ and $x_2 := 75$. Thus $L_1 := \{(59, \pm 54)\}$ and $L_2 := \{(75, \pm 1)\}$. The Miller algorithm gives $h_{140,A}((59, 54)) = 114\theta + 109$, $h_{140,A}((59, -54)) = 25\theta + 109$, $h_{140,A}((75, 1)) = 112\theta + 22$ and $h_{140,A}((75, -1)) = 27\theta + 22$. Therefore we obtain the desired answer $Q := (59, -54)$.

We observe an example for a non-generator. Put $B := 5A$ and $v := 56\theta + 55$ whose orders are both 7. There are five points $Q_n := (83, 55) + n(69, 11) \in G_1$, where $0 \le n < 5$, satisfying $e_{140}(B, Q_n) = v$. Although the paring values are equal, the algorithm requires correct input from FEI, which are different for each n. For example, the algorithm returns unique point Q_0 for input $(4\theta + 135, B)$, whereas it returns unique point Q_1 for input $(98\theta + 41, B)$. It is a role of FEI to provide a correct value to Algorithm 4.1.

5. The case of Odd Embedding Degrees

In this section, we present a probabilistic polynomial time algorithm for the Miller inversion in case that the embedding degree $k > 1$ is odd. As was in the previous section, q is a power of an odd prime p and $E/\mathbf{F}_q$ denotes an elliptic curve given by the Weierstrass model. Throughout the section, we restrict ourselves to the case that l is a prime divisor of $\#E(\mathbf{F}_q)$ which is different from p. Put $G_1 := E[l] \cap E(\mathbf{F}_q)$ and $G_0 := \{ P \in E[l] : \sigma_q P = qP \}$. Let k be the embedding degree of l for $E/\mathbf{F}_q$. Let $e \ge 3$ be any prime factor of k. Note that the conditions $k > 1$ and $k \equiv 1 \bmod 2$ ensure that such e certainly exists. For simplicity, we put $r := q^k$, $s := q^{k/e}$ and

$$t_n := \sum_{i=1}^{n} s^i$$

for $n \ge 1$. Denote the e-th cyclotomic polynomial by Φ_e. Minimality of k and primality of l and e imply $l \mid \Phi_e(s)$. For any points P and R on E, we denote by V_P and $L_{P,R}$ the unique normalized rational function with a divisor

$$\begin{aligned} \mathrm{div}(L_{P,R}) &= [P] + [R] + [-P-R] - 3[\mathcal{O}], \\ \mathrm{div}(V_P) &= [P] + [-P] - 2[\mathcal{O}], \end{aligned}$$

respectively. Then

$$\mathrm{ord}_{\mathcal{O}} L_{P,R} = \begin{cases} -3 & (P \ne \mathcal{O},\ R \ne \mathcal{O},\ P+R \ne \mathcal{O}), \\ 0 & (P = R = \mathcal{O}), \\ -2 & (\text{otherwise}), \end{cases} \tag{5.1}$$

$$\mathrm{ord}_{\mathcal{O}} V_P = \begin{cases} -2 & (P \ne \mathcal{O}), \\ 0 & (P = \mathcal{O}). \end{cases} \tag{5.2}$$

Since they are regular outside of $\{\mathcal{O}\}$, they are in fact elements of the coordinate ring of E.

Lemma 5.1. *Let A be a point of $E[l]$ different from $\mathcal{O}$. For $n \in \mathbf{N}$, we have the followings:*

(i) $t_n A \ne \mathcal{O}$ *for* $1 \le n \le e-1$.

(ii) $\mathrm{ord}_{\mathcal{O}} L_{s^n A,\, t_{n-1} A} = -3$ *for* $2 \le n \le e-1$.

(iii) $\mathrm{ord}_{\mathcal{O}} V_{t_n A} = -2$ *for* $1 \le n \le e-1$.

Proof. (i) Suppose $t_n A = \mathcal{O}$. Since $\mathcal{O} = (s-1) t_n A = s(s-1) \sum_{i=0}^{n-1} s^i A = s(s^n - 1)A$, we have $l \mid s^n - 1$. By the minimality of k, we see $k \le \frac{k}{e} n$, i.e., $n \ge e$. Thus $t_n A \ne \mathcal{O}$ for $1 \le n < e$. For assertion (ii), note $s^n A \ne \mathcal{O}$ and $t_{n-1} A \ne \mathcal{O}$. Moreover $s^n A + t_{n-1} A = t_n A \ne \mathcal{O}$ because of $n \le e-1$. By (5.1), we see (ii) holds. The assertion (iii) is an immediate consequence of (i) and (5.2). □

By induction on n, we have

$$h_{t_n,A} = \frac{\prod_{i=2}^{n} L_{s^iA,\,t_{i-1}A}}{\prod_{i=2}^{n} V_{t_iA}} \prod_{i=1}^{n} h_{s^i,A}. \tag{5.3}$$

(We understand that an empty product is 1.)

Before we present our algorithm, we discuss a data structure for the coordinate ring of E. Recall that E is given by the Wierestrass model. Therefore $\mathbf{F}_r[\xi, \eta]$ is a free $\mathbf{F}_r[\xi]$ module of rank 2 whose generator is 1 and η. This means that every element $f \in \mathbf{F}_r[\xi, \eta]$ is uniquely represented as $f = f_1 + f_2\eta$ with $f_1,\ f_2 \in \mathbf{F}_r[\xi]$. In case $\mathrm{ord}_{\mathcal{O}} f \geq -n$, it holds that $\deg f_1 \leq n/2$ and that $\deg f_2 \leq (n-3)/2$. With this representation, space complexity for $f \in \mathbf{F}_r[\xi, \eta]$ is $O((n+1)\log r)$ bit if $\mathrm{ord}_{\mathcal{O}} f = -n$. Time complexity of one arithmetic operation between f and $g \in \mathbf{F}_r[\xi, \eta]$ is bounded by $O((\max(-\mathrm{ord}_{\mathcal{O}} f, -\mathrm{ord}_{\mathcal{O}} g)+1)^2 (\log r)^2)$ bit operations.

Algorithm 5.2.
Input: $d \in \mathbf{N}$ satisfying $l \mid d$ and $d \mid \Phi_e(s)$,
$v \in \mathbf{F}_r$,
$A \in G_0 - \{\mathcal{O}\}$.
Output: $Q \in G_1 - \{\mathcal{O}\}$ satisfying $h_{d,A}(Q) = v$ if such Q exists. Otherwise, **nil**.
Procedure:

```
 1: c := Φ_e(s) ;
 2: u := (v^{c/d})^c ;
 3: δ := 1 ; ν := 1 ;
 4: B := A ;
 5: for (m := 0 ; m < e ; m := m+1) {
 6:    ν := ν*V_B ; // here, B = σ_s^m(A)
 7:    S := B ;
 8:    T := B ;
 9:    for (i := 2 ; i < e ; i := i+1) {
10:       // here, T = t_{i-1} σ_s^m(A)
11:       S := sS ; // S = s^i σ_s^m(A)
12:       ν := ν*L_{S,T} ;
13:       T := S+T ; // T = t_i σ_s^m(A)
14:       δ := δ*V_T ;
15:    }
16:    B := σ_s(B) ;
17: }
18: Build a set Λ := {Q ∈ E(F_q) : ν(Q) = uδ(Q)} // Here, 0 ≤ #Λ ≤ 3e^2.
19: for (Q ∈ Λ) {
20:    if lQ = O and h_{d,A}(Q) = v then return Q ;
21: }
22: return nil ;
```

Theorem 5.3. *Algorithm 5.2 returns a correct result with $O(k^6(\log q)^3)$ bit operations in average.*

Proof. A proof for correctness is similar to the proof of Theorem 4.2. Suppose that there exists $Q \in G_1 - \{\mathcal{O}\}$ satisfying $h_{d,A}(Q) = v$. Since $l \mid d \mid c$, we have $h_{c,A} = h_{d,A}^{c/d}$ and

$$h_{c,A} = V_A h_{t_{e-1},A}. \tag{5.4}$$

By (5.3), it holds that

$$h_{c,A} = V_A \cdot \frac{\prod_{i=2}^{e-1} L_{s^iA,\, t_{i-1}A}}{\prod_{i=2}^{e-1} V_{t_iA}} \prod_{i=1}^{e-1} h_{s^i,A}.$$

By Granger et al.[10, Theorem 2], we have $h_{s^i,A}(Q) \in \mu_l \subset \mu_c$ as before. Thus evaluation of (5.4) at Q followed by $c = \sum_{m=0}^{e-1} s^m$ powering yields

$$(v^{c/d})^c = \prod_{m=0}^{e-1} \left(V_A(Q) \frac{\prod_{i=2}^{e-1} L_{s^iA,\, t_{i-1}A}(Q)}{\prod_{i=2}^{e-1} V_{t_iA}(Q)} \right)^{s^m}. \tag{5.5}$$

However, computing s^m powering of an element in $\mathbf{F}_r$ is nothing but applying σ_s^m. Since E is defined over $\mathbf{F}_q$ and $Q \in E(\mathbf{F}_q)$,

$$u = \prod_{m=0}^{e-1} \left(\frac{V_{\sigma_s^m A}(Q) \prod_{i=2}^{e-1} L_{s^i\sigma_s^m A,\, t_{i-1}\sigma_s^m A}(Q)}{\prod_{i=2}^{e-1} V_{t_i\sigma_s^m A}(Q)} \right) = \frac{v(Q)}{\delta(Q)}. \tag{5.6}$$

This proves that $(v - u\delta)(Q) = 0$. Using Lemma 5.1, we have

$$\mathrm{ord}_{\mathcal{O}} v = -e(2+3(e-2)) \text{ and } \mathrm{ord}_{\mathcal{O}} \delta = -2e(e-2). \tag{5.7}$$

Note $\mathrm{ord}_{\mathcal{O}} v < \mathrm{ord}_{\mathcal{O}} \delta$ for $e \geq 3$. Therefore $\mathrm{ord}_{\mathcal{O}}(v - u\delta) = -e(3e-4) \neq 0$. In particular, $v - u\delta \in \mathbf{F}_r[\xi, \eta]$ is not a constant and has $e(3e-4)$ zeros with counting multiplicities. Thus $\#\Lambda \leq e(3e-4) \leq 3e^2$. This completes the proof of correctness.

Now we analyze computational complexity of Algorithm 5.2. It is clear that time complexities of Steps 1 and 2 are bounded by $O(e(\log r)(\log s)) = O(k^2(\log q)^3)$ and $O((\log c^2)(\log r)) = O(k^2(\log q)^2)$ bit operations, respectively. In inside of the loop begining at Step 5, the orders of pole $-\mathrm{ord}_{\mathcal{O}} v$ and $-\mathrm{ord}_{\mathcal{O}} \delta$ increase monotonically. By (5.7), number of arithmetic operation on $\mathbf{F}_r$ during Steps 10–14 are bounded by $O(e^2 + \log s)$. Therefore the number of bit operations performed for the loop beggining at Step 5 is $O(e^2(k^2 + \log s)(\log r)^2)$, which is bounded by $O(k^5(k + \log q)(\log q)^2)$. In order to find zeros of $v - u\delta$, define f_1, $f_2 \in \mathbf{F}_r[\xi]$ by $f_1 + \eta f_2 = v - u\delta$. Let

$$Y^2 + a_1XY + a_3Y - X^3 - a_2X^2 - a_4X - a_6 = 0$$

be the Wierestrass model defining E. Put

$$\Xi_1 := \{ x \in \mathbf{F}_q : f_2(x) = 0 \},$$

$$F := f_1^2 - a_1\xi f_1 f_2 - a_3 f_1 f_2 - f_2^2(\xi^3 + a_2\xi^2 + a_4\xi + a_6) \in \mathbf{F}_r[\xi],$$

$$\Xi_2 := \{ x \in \mathbf{F}_q : F(x) = 0 \}.$$

Then, $\xi(Q) \in \Xi_1 \cup \Xi_2$. Letting $N := -\mathrm{ord}_{\mathcal{O}}(v - u\delta) = e(3e-4)$, we have $\deg f_1 \leq N/2$ and $\deg f_2 \leq (N-3)/2$. Therefore $\deg F \leq N$. A probabilistic factoring algorithm for polynomials over finite fields produces Ξ_1 and Ξ_2 which amounts to $O(e^3 \log r)$ arithmetic operations

in $\mathbf{F}_r$ (with naive multiplications) in average. (See e.g. von zur Gathen and Gerhard[19, Theorem 14.14].) For each candedate x of $\xi(Q)$, we need only $O(1)$ arithmetic operations and square root computations on $\mathbf{F}_q$ to recover Q (if any). Thus overall time complexity is bounded by $O(k^6(\log q)^3)$ bit operations in average. □

Remark 5.4. Observe that $v-u\delta$ is σ_s invariant. This implies $v-u\delta \in \mathbf{F}_s[\xi, \eta]$ and f_1, $f_2 \in \mathbf{F}_s[\xi]$. Implementation to take advantage of this fact needs to map an element $x \in \mathbf{F}_r$ satisfying $\sigma_s(x)=x$ to $\mathbf{F}_s$. This is trivial in case of $s=q=p$. However, in general, time complexity of such conversion depends on implementation of field extensions, which is out of scope of this article. We also note that we do not need complete factorization. Let ψ be $X^q \bmod f_2(X)$ computed in $\mathbf{F}_s[X]/\langle f_2(X)\rangle$. To obtain Ξ_1, it suffices to apply equal degree factorization to $\gcd(f_2, \psi)$. The same method applies to F. Finally, polynomial factorizations are only the places which make our algorithm probabilistic.

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