

# L-functions

(PARI-GP version 2.13.3)

## Characters

A character on the abelian group  $G = \sum_{j \leq k} (\mathbf{Z}/d_j \mathbf{Z}) \cdot g_j$ , e.g. from `znstar(q,1)`  $\leftrightarrow (\mathbf{Z}/q\mathbf{Z})^*$  or `bnrinit`  $\leftrightarrow \text{Cl}_f(K)$ , is coded by  $\chi = [c_1, \dots, c_k]$  such that  $\chi(g_j) = e(c_j/d_j)$ . Our  $L$ -functions consider the attached *primitive* character.

Dirichlet characters  $\chi_q(m, \cdot)$  in Conrey labelling system are alternatively concisely coded by `Mod(m,q)`. Finally, a quadratic character  $(D/\cdot)$  can also be coded by the integer  $D$ .

## L-function Constructors

An `Ldata` is a GP structure describing the functional equation for  $L(s) = \sum_{n \geq 1} a_n n^{-s}$ .

- Dirichlet coefficients given by closure  $a : N \mapsto [a_1, \dots, a_N]$ .
- Dirichlet coefficients  $a^*(n)$  for dual  $L$ -function  $L^*$ .
- Euler factor  $A = [a_1, \dots, a_d]$  for  $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$ ,
- classical weight  $k$  (values at  $s$  and  $k - s$  are related),
- conductor  $N$ ,  $\Lambda(s) = N^{s/2} \gamma_A(s)$ ,
- root number  $\varepsilon$ ;  $\Lambda(a, k - s) = \varepsilon \Lambda(a^*, s)$ .
- polar part: list of  $[\beta, P_\beta(x)]$ .

An `Linit` is a GP structure containing an `Ldata`  $L$  and an evaluation *domain* fixing a maximal order of derivation  $m$  and bit accuracy (`realbitprecision`), together with complex ranges

- for  $L$ -function:  $R = [c, w, h]$  (coding  $|\Re z - c| \leq w, |\Im z| \leq h$ ); or  $R = [w, h]$  (for  $c = k/2$ ); or  $R = [h]$  (for  $c = k/2, w = 0$ ).
- for  $\theta$ -function:  $T = [\rho, \alpha]$  (for  $|t| \geq \rho, |\arg t| \leq \alpha$ ); or  $T = \rho$  (for  $\alpha = 0$ ).

### Ldata constructors

Riemann $\zeta$	<code>lfuncreate(1)</code>
Dirichlet for quadratic char. $(D/\cdot)$	<code>lfuncreate(D)</code>
Dirichlet series $L(\chi_q(m, \cdot), s)$	<code>lfuncreate(Mod(m,q))</code>
Dedekind $\zeta_K, K = \mathbf{Q}[x]/(T)$	<code>lfuncreate(bnf), lfuncreate(T)</code>
Hecke for $\chi \bmod \mathfrak{f}$	<code>lfuncreate([bnr, \chi])</code>
Artin $L$ -function	<code>lfunartin(nf, gal, M, n)</code>
Lattice $\Theta$ -function	<code>lfunqf(Q)</code>
From eigenform $F$	<code>lfunmf(F)</code>
Quotients of Dedekind $\eta: \prod_i \eta(m_{i,1} \cdot \tau)^{m_{i,2}}$	<code>lfunetaquo(M)</code>
$L(E, s), E$ elliptic curve	<code>E = ellinit(...)</code>
$L(\text{Sym}^m E, s), E$ elliptic curve	<code>lfunsympow(E, m)</code>
genus 2 curve, $y^2 + xQ = P$	<code>lfungenus2([P, Q])</code>
dual $L$ function $\hat{L}$	<code>lfundual(L)</code>
$L_1 \cdot L_2$	<code>lfunmul(L1, L2)</code>
$L_1/L_2$	<code>lfundiv(L1, L2)</code>
$L(s - d)$	<code>lfunshift(L, d)</code>
$L(s) \cdot L(s - d)$	<code>lfunshift(L, d, 1)</code>
twist by Dirichlet character	<code>lfuntwist(L, \chi)</code>
low-level constructor	<code>lfuncreate([a, a*, A, k, N, eps, poles])</code>
check functional equation (at $t$ )	<code>lfuncheckfeq(L, {t})</code>

### Linit constructors

initialize for $L$	<code>lfuninit(L, R, {m = 0})</code>
initialize for $\theta$	<code>lfunthetainit(L, {T = 1}, {m = 0})</code>
cost of <code>lfuninit</code>	<code>lfuncost(L, R, {m = 0})</code>
cost of <code>lfunthetainit</code>	<code>lfunthetacost(L, T, {m = 0})</code>
Dedekind $\zeta_L, L$ abelian over a subfield	<code>lfunabelianreinit</code>

## L-functions

$L$  is either an `Ldata` or an `Linit` (more efficient for many values).

### Evaluate

$L^{(k)}(s)$	<code>lfun(L, s, {k = 0})</code>
$\Lambda^{(k)}(s)$	<code>lfunlambda(L, s, {k = 0})</code>
$\theta^{(k)}(t)$	<code>lfuntheta(L, t, {k = 0})</code>
generalized Hardy $Z$ -function at $t$	<code>lfunhardy(L, t)</code>

### Zeros

order of zero at $s = k/2$	<code>lfunorderzero(L, {m = -1})</code>
zeros $s = k/2 + it, 0 \leq t \leq T$	<code>lfunzeros(L, T, {h})</code>

### Dirichlet series and functional equation

$[a_n: 1 \leq n \leq N]$	<code>lfunan(L, N)</code>
conductor $N$ of $L$	<code>lfunconductor(L)</code>
root number and residues	<code>lfunrootres(L)</code>

### G-functions

Attached to inverse Mellin transform for $\gamma_A(s), A = [a_1, \dots, a_d]$ .	
initialize for $G$ attached to $A$	<code>gammamellinininit(A)</code>
$G^{(k)}(t)$	<code>gammamellinininv(G, t, {k = 0})</code>
asypm. expansion of $G^{(k)}(t)$	<code>gammamellininvasymp(A, n, {k = 0})</code>

Based on an earlier version by Joseph H. Silverman

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