Modular forms, modular symbols

(PARI-GP version 2.16.2)

Modular Forms

Dirichlet characters

Characters are encoded in three different ways:

- a t_INT $D \equiv 0, 1 \bmod 4$: the quadratic character $(D/\cdot)$;
- a t_INTMOD Mod(m, q), $m \in (\mathbf{Z}/q)^*$ using a canonical bijection with the dual group (the Conrey character $\chi_q(m, \cdot)$);
- a pair $[G, chi]$, where $G = znstar(q, 1)$ encodes $(\mathbf{Z}/q\mathbf{Z})^* = \sum_{j \le k} (\mathbf{Z}/d_j\mathbf{Z}) \cdot g_j$ and the vector $chi = [c_1, \dots, c_k]$ encodes the character such that $\chi(g_j) = e(c_j/d_j)$.

initialize $G = (\mathbf{Z}/q\mathbf{Z})^*$	`G = znstar`$(q, 1)$
convert datum D to $[G, \chi]$	`znchar`(D)
Galois orbits of Dirichlet characters	`chargalois`(G)

Spaces of modular forms

Arguments of the form $[N, k, \chi]$ give the level weight and nebentypus χ; χ can be omitted: $[N, k]$ means trivial χ.

initialize $S_k^{\text{new}}(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 0)$
initialize $S_k(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 1)$
initialize $S_k^{\text{old}}(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 2)$
initialize $E_k(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi], 3)$
initialize $M_k(\Gamma_0(N), \chi)$	`mfinit`$([N, k, \chi])$
find eigenforms	`mfsplit`(M)
statistics on self-growing caches	`getcache()`

We let $M =$ `mfinit`$(\dots)$ denote a modular space.

describe the space M	`mfdescribe`(M)
recover (N, k, χ)	`mfparams`(M)
...the space identifier (0 to 4)	`mfspace`(M)
...the dimension of M over $\mathbf{C}$	`mfdim`(M)
...a $\mathbf{C}$-basis (f_i) of M	`mfbasis`(M)
...a basis (F_j) of eigenforms	`mfeigenbasis`(M)
...polynomials defining $\mathbf{Q}(\chi)(F_j)/\mathbf{Q}(\chi)$	`mffields`(M)
matrix of Hecke operator T_n on (f_i)	`mfheckemat`(M, n)
eigenvalues of w_Q	`mfatkineigenvalues`(M, Q)
basis of period poynomials for weight k	`mfperiodpolbasis`(k)
basis of the Kohnen $+$-space	`mfkohnenbasis`(M)
...new space and eigenforms	`mfkohneneigenbasis`(M, b)
isomorphism $S_k^+(4N, \chi) \to S_{2k-1}(N, \chi^2)$	`mfkohnenbijection`(M)

Useful data can also be obtained a priori, without computing a complete modular space:

dimension of $S_k^{\text{new}}(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi])$
dimension of $S_k(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 1)$
dimension of $S_k^{\text{old}}(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 2)$
dimension of $M_k(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 3)$
dimension of $E_k(\Gamma_0(N), \chi)$	`mfdim`$([N, k, \chi], 4)$
Sturm's bound for $M_k(\Gamma_0(N), \chi)$	`mfsturm`(N, k)

$\Gamma_0(N)$ cosets

list of right $\Gamma_0(N)$ cosets	`mfcosets`(N)
identify coset a matrix belongs to	`mftocoset`

Cusps

a cusp is given by a rational number or `oo`.

lists of cusps of $\Gamma_0(N)$	`mfcusps`(N)
number of cusps of $\Gamma_0(N)$	`mfnumcusps`(N)
width of cusp c of $\Gamma_0(N)$	`mfcuspwidth`(N, c)
is cusp c regular for $M_k(\Gamma_0(N), \chi)$?	`mfcuspisregular`$([N, k, \chi], c)$

Create an individual modular form

Besides `mfbasis` and `mfeigenbasis`, an individual modular form can be identified by a few coefficients.

modular form from coefficients	`mftobasis`$(\texttt{mf}, vec)$

There are also many predefined ones:

Eisenstein series E_k on $Sl_2(\mathbf{Z})$	`mfEk`(k)
Eisenstein-Hurwitz series on $\Gamma_0(4)$	`mfEH`(k)
unary θ function (for character ψ)	`mfTheta`$(\{\psi\})$
Ramanujan's Δ	`mfDelta()`
$E_k(\chi)$	`mfeisenstein`(k, χ)
$E_k(\chi_1, \chi_2)$	`mfeisenstein`(k, χ_1, χ_2)
eta quotient $\prod_i \eta(a_{i,1} \cdot z)^{a_{i,2}}$	`mffrometaquo`(a)
newform attached to ell. curve $E/\mathbf{Q}$	`mffromell`(E)
identify an L-function as a eigenform	`mffromlfun`(L)
θ function attached to $Q > 0$	`mffromqf`(Q)
trace form in $S_k^{\text{new}}(\Gamma_0(N), \chi)$	`mftraceform`$([N, k, \chi])$
trace form in $S_k(\Gamma_0(N), \chi)$	`mftraceform`$([N, k, \chi], 1)$

Operations on modular forms

In this section, f, g and the $F[i]$ are modular forms

$f \times g$	`mfmul`(f, g)
f/g	`mfdiv`(f, g)
f^n	`mfpow`(f, n)
$f(q)/q^v$	`mfshift`(f, v)
$\sum_{i \le k} \lambda_i F[i]$, $L = [\lambda_1, \dots, \lambda_k]$	`mflinear`(F, L)
$f = g$?	`mfisequal(f,g)`
expanding operator $B_d(f)$	`mfbd`(f, d)
Hecke operator $T_n f$	`mfhecke`(mf, f, n)
initialize Atkin–Lehner operator w_Q	`mfatkininit`(mf, Q)
...apply w_Q to f	`mfatkin`(w_Q, f)
twist by the quadratic char $(D/\cdot)$	`mftwist`(f, D)
derivative wrt. $q \cdot d/dq$	`mfderiv`(f)
see f over an absolute field	`mfreltoabs`(f)
Serre derivative $\left(q \cdot \frac{d}{dq} - \frac{k}{12} E_2\right) f$	`mfderivE2`(f)
Rankin-Cohen bracket $[f, g]_n$	`mfbracket`(f, g, n)
Shimura lift of f for discriminant D	`mfshimura`(mf, f, D)

Properties of modular forms

In this section, $f = \sum_n f_n q^n$ is a modular form in some space M with parameters N, k, χ.

describe the form f	`mfdescribe`(f)
(N, k, χ) for form f	`mfparams`(f)
the space identifier (0 to 4) for f	`mfspace`(mf, f)
$[f_0, \dots, f_n]$	`mfcoefs`(f, n)
f_n	`mfcoef`(f, n)
is f a CM form?	`mfisCM`(f)
is f an eta quotient?	`mfisetaquo`(f)
Galois rep. attached to all $(1, \chi)$ eigenforms	`mfgaloistype`(M)
...single eigenform	`mfgaloistype`(M, F)
...as a polynomial fixed by Ker ρ_F	`mfgaloisprojrep`(M, F)
decompose f on `mfbasis`(M)	`mftobasis`(M, f)
smallest level on which f is defined	`mfconductor`(M, f)
decompose f on $\oplus S_k^{\text{new}}(\Gamma_0(d))$, $d \mid N$	`mftonew`(M, f)
valuation of f at cusp c	`mfcuspval`(M, f, c)
expansion at ∞ of $f \mid_k \gamma$	`mfslashexpansion`(M, f, γ, n)
n-Taylor expansion of f at i	`mftaylor`(f, n)
all rational eigenforms matching criteria	`mfeigensearch`
...forms matching criteria	`mfsearch`

Forms embedded into C

Given a modular form f in $M_k(\Gamma_0(N), \chi)$ its field of definition $Q(f)$ has $n = [Q(f) : Q(\chi)]$ embeddings into the complex numbers. If $n = 1$, the following functions return a single answer, attached to the canonical embedding of f in $\mathbf{C}[[q]]$; else a vector of n results, corresponding to the n conjugates of f.

complex embeddings of $Q(f)$	`mfembed`(f)
... embed coefs of f	`mfembed`(f, v)
evaluate f at $\tau \in \mathcal{H}$	`mfeval`(f, τ)
L-function attached to f	`lfunmf`(mf, f)
...eigenforms of new space M	`lfunmf`(M)

Periods and symbols

The functions in this section depend on $[Q(f) : Q(\chi)]$ as above.

initialize symbol fs attached to f	`mfsymbol`(M, f)
evaluate symbol fs on path p	`mfsymboleval`(fs, p)
Petersson product of f and g	`mfpetersson`(fs, gs)
period polynomial of form f	`mfperiodpol`(M, fs)
period polynomials for eigensymbol FS	`mfmanin`(FS)

Modular Symbols

Let $G = \Gamma_0(N)$, $V_k = \mathbf{Q}[X, Y]_{k-2}$, $L_k = \mathbf{Z}[X, Y]_{k-2}$ and $\Delta = \text{Div}^0(\mathbf{P}^1(\mathbf{Q}))$. An element of Δ is a *path* between cusps of $X_0(N)$ via the identification $[b] - [a] \to$ path from a to b, coded by the pair $[a, b]$ where a, b are rationals or `oo` $= (1 : 0)$.

Let $\mathbf{M}_k(G) = \text{Hom}_G(\Delta, V_k) \simeq H_c^1(X_0(N), V_k)$; an element of $\mathbf{M}_k(G)$ is a V_k-valued *modular symbol*. There is a natural decomposition $\mathbf{M}_k(G) = \mathbf{M}_k(G)^+ \oplus \mathbf{M}_k(G)^-$ under the action of the $*$ involution, induced by complex conjugation. The `msinit` function computes either $\mathbf{M}_k$ ($\varepsilon = 0$) or its $\pm$-parts ($\varepsilon = \pm 1$) and fixes a minimal set of $\mathbf{Z}[G]$-generators (g_i) of Δ.

initialize $M = \mathbf{M}_k(\Gamma_0(N))^\varepsilon$	`msinit`$(N, k, \{\varepsilon = 0\})$
the level M	`msgetlevel`(M)
the weight k	`msgetweight`(M)
the sign ε	`msgetsign`(M)
Farey symbol attached to G	`mspolygon`(M)
...attached to $H < G$	`msfarey`(F, inH)
$H \backslash G$ and right G-action	`mscosets`$(genG, inH)$
$\mathbf{Z}[G]$-generators (g_i) and relations for Δ	`mspathgens`(M)
decompose $p = [a, b]$ on the (g_i)	`mspathlog`(M, p)

Create a symbol

Eisenstein symbol attached to cusp c	`msfromcusp`(M, c)
cuspidal symbol attached to $E/\mathbf{Q}$	`msfromell`(E)
symbol having given Hecke eigenvalues	`msfromhecke`$(M, v, \{H\})$
is s a symbol ?	`msissymbol`(M, s)

Operations on symbols

the list of all $s(g_i)$	`mseval`(M, s)
evaluate symbol s on path $p = [a, b]$	`mseval`(M, s, p)
Petersson product of s and t	`mspetersson`(M, s, t)

Operators on subspaces

An operator is given by a matrix of a fixed $\mathbf{Q}$-basis. H, if given, is a stable $\mathbf{Q}$-subspace of $\mathbf{M}_k(G)$: operator is restricted to H.

matrix of Hecke operator T_p or U_p	`mshecke`$(M, p, \{H\})$
matrix of Atkin-Lehner w_Q	`msatkinlehner`$(M, Q\{H\})$
matrix of the $*$ involution	`msstar`$(M, \{H\})$

Subspaces

A subspace is given by a structure allowing quick projection and restriction of linear operators. Its fist component is a matrix with integer coefficients whose columns for a $\mathbf{Q}$-basis. If H is a Hecke-stable subspace of $M_k(G)^+$ or $M_k(G)^-$, it can be split into a direct sum of Hecke-simple subspaces. To a simple subspace corresponds a single normalized newform $\sum_n a_n q^n$.

cuspidal subspace $S_k(G)^\varepsilon$	`mscuspidal`(M)
Eisenstein subspace $E_k(G)^\varepsilon$	`mseisenstein`(M)
new part of $S_k(G)^\varepsilon$	`msnew`(M)
split H into simple subspaces (of dim $\leq d$)	`mssplit`$(M, H, \{d\})$
dimension of a subspace	`msdim`(M)
$(a_1, \ldots, a_B)$ for attached newform	`msqexpansion`$(M, H, \{B\})$
$\mathbf{Z}$-structure from $H^1(G, L_k)$ on subspace A	`mslattice`(M, A)

Overconvergent symbols and p-adic L functions

Let M be a full modular symbol space given by `msinit` and p be a prime. To a classical modular symbol ϕ of level N ($v_p(N) \leq 1$), which is an eigenvector for T_p with nonzero eigenvalue a_p, we can attach a p-adic L-function L_p. The function L_p is defined on continuous characters of $\mathrm{Gal}(\mathbf{Q}(\mu_{p^\infty})/\mathbf{Q})$; in GP we allow characters $\langle\chi\rangle^{s_1}\tau^{s_2}$, where (s_1, s_2) are integers, τ is the Teichmüller character and χ is the cyclotomic character.

The symbol ϕ can be lifted to an *overconvergent* symbol Φ, taking values in spaces of p-adic distributions (represented in GP by a list of moments modulo p^n).

`mspadicinit` precomputes data used to lift symbols. If *flag* is given, it speeds up the computation by assuming that $v_p(a_p) = 0$ if *flag* $= 0$ (fastest), and that $v_p(a_p) \geq$ *flag* otherwise (faster as *flag* increases).

`mspadicmoments` computes distributions mu attached to Φ allowing to compute L_p to high accuracy.

initialize Mp to lift symbols	`mspadicinit`$(M, p, n, \{\mathit{flag}\})$
lift symbol ϕ	`mstooms`(Mp, ϕ)
eval overconvergent symbol Φ on path p	`msomseval`(Mp, Φ, p)
mu for p-adic L-functions	`mspadicmoments`$(Mp, S, \{D = 1\})$
$L_p^{(r)}(\chi^s)$, $s = [s_1, s_2]$	`mspadicL`$(mu, \{s = 0\}, \{r = 0\})$
$\hat{L}_p(\tau^i)(x)$	`mspadicseries`$(mu, \{i = 0\})$

Send comments and corrections to ⟨Karim.Belabas@math.u-bordeaux.fr⟩