

# L-functions

(PARI-GP version 2.17.0)

## Characters

A character on the abelian group  $G = \sum_{j \leq k} (\mathbf{Z}/d_j \mathbf{Z}) \cdot g_j$ , e.g. from `znstar(q,1)`  $\leftrightarrow (\mathbf{Z}/q\mathbf{Z})^*$  or `bnrinit`  $\leftrightarrow \text{Cl}_f(K)$ , is coded by  $\chi = [c_1, \dots, c_k]$  such that  $\chi(g_j) = e(c_j/d_j)$ . Our  $L$ -functions consider the attached *primitive* character.

Dirichlet characters  $\chi_q(m, \cdot)$  in Conrey labelling system are alternatively concisely coded by `Mod(m,q)`. Finally, a quadratic character  $(D/\cdot)$  can also be coded by the integer  $D$ .

## L-function Constructors

An `Ldata` is a GP structure describing the functional equation for  $L(s) = \sum_{n \geq 1} a_n n^{-s}$ .

- Dirichlet coefficients given by closure  $a : N \mapsto [a_1, \dots, a_N]$ .
- Dirichlet coefficients  $a^*(n)$  for dual  $L$ -function  $L^*$ .
- Euler factor  $A = [a_1, \dots, a_d]$  for  $\gamma_A(s) = \prod_i \Gamma_{\mathbf{R}}(s + a_i)$ ,
- classical weight  $k$  (values at  $s$  and  $k - s$  are related),
- conductor  $N$ ,  $\Lambda(s) = N^{s/2} \gamma_A(s)$ ,
- root number  $\varepsilon$ ;  $\Lambda(a, k - s) = \varepsilon \Lambda(a^*, s)$ .
- polar part: list of  $[\beta, P_\beta(x)]$ .

An `Linit` is a GP structure containing an `Ldata`  $L$  and an evaluation *domain* fixing a maximal order of derivation  $m$  and bit accuracy (`realbitprecision`), together with complex ranges

- for  $L$ -function:  $R = [c, w, h]$  (coding  $|\Re z - c| \leq w$ ,  $|\Im z| \leq h$ ); or  $R = [w, h]$  (for  $c = k/2$ ); or  $R = [h]$  (for  $c = k/2$ ,  $w = 0$ ).
- for  $\theta$ -function:  $T = [\rho, \alpha]$  (for  $|t| \geq \rho$ ,  $|\arg t| \leq \alpha$ ); or  $T = \rho$  (for  $\alpha = 0$ ).

### Ldata constructors

Riemann $\zeta$	<code>lfuncreate(1)</code>
Dirichlet for quadratic char. $(D/\cdot)$	<code>lfuncreate(D)</code>
Dirichlet series $L(\chi_q(m, \cdot), s)$	<code>lfuncreate(Mod(m,q))</code>
Dedekind $\zeta_K$ , $K = \mathbf{Q}[x]/(T)$	<code>lfuncreate(bnf)</code> , <code>lfuncreate(T)</code>
Hecke for $\chi \bmod \mathfrak{f}$	<code>lfuncreate([bnr, \chi])</code>
Artin $L$ -function	<code>lfunartin(nf, gal, M, n)</code>
Lattice $\Theta$ -function	<code>lfunqf(Q)</code>
From eigenform $F$	<code>lfunmf(F)</code>
Quotients of Dedekind $\eta: \prod_i \eta(m_{i,1} \cdot \tau)^{m_{i,2}}$	<code>lfunetaquo(M)</code>
$L(E, s)$ , $E$ elliptic curve	<code>E = ellinit(...)</code>
$L(\text{Sym}^m E, s)$ , $E$ elliptic curve	<code>lfunsympow(E, m)</code>
Genus 2 curve, $y^2 + xQ = P$	<code>lfungenus2([P,Q])</code>
Hypergeometric motive $H(a, b; t)$	<code>lfunhgm(hgminit(a,b), t)</code>

dual $L$ function $\hat{L}$	<code>lfundual(L)</code>
$L_1 \cdot L_2$	<code>lfunmul(L1, L2)</code>
$L_1/L_2$	<code>lfundiv(L1, L2)</code>
$L(s - d)$	<code>lfunshift(L, d)</code>
$L(s) \cdot L(s - d)$	<code>lfunshift(L, d, 1)</code>
twist by Dirichlet character	<code>lfuntwist(L, \chi)</code>

low-level constructor	<code>lfuncreate([a, a*, A, k, N, eps, poles])</code>
check functional equation (at $t$ )	<code>lfuncheckfeq(L, {t})</code>
parameters $[N, k, A]$	<code>lfunparams(L)</code>

### Linit constructors

initialize for $L$	<code>lfuninit(L, R, {m = 0})</code>
initialize for $\theta$	<code>lfunthetainit(L, {T = 1}, {m = 0})</code>
cost of <code>lfuninit</code>	<code>lfuncost(L, R, {m = 0})</code>
cost of <code>lfunthetainit</code>	<code>lfunthetacost(L, T, {m = 0})</code>
Dedekind $\zeta_L$ , $L$ abelian over a subfield	<code>lfunabelianreinit</code>

## L-functions

$L$  is an `Ldata` or an `Linit` (more efficient for many values).

### Evaluate

$L^{(k)}(s)$	<code>lfun(L, s, {k = 0})</code>
$\Lambda^{(k)}(s)$	<code>lfunlambda(L, s, {k = 0})</code>
$\theta^{(k)}(t)$	<code>lfuntheta(L, t, {k = 0})</code>

generalized Hardy  $Z$ -function at  $t$

`lfunhardy(L, t)`

### Zeros

order of zero at $s = k/2$	<code>lfunorderzero(L, {m = -1})</code>
zeros $s = k/2 + it$ , $0 \leq t \leq T$	<code>lfunzeros(L, T, {h})</code>

### Dirichlet series and functional equation

$[a_n: 1 \leq n \leq N]$	<code>lfunan(L, N)</code>
Euler factor at $p$	<code>lfuneuler(L, p)</code>
conductor $N$ of $L$	<code>lfunconductor(L)</code>
root number and residues	<code>lfunrootres(L)</code>

### G-functions

Attached to inverse Mellin transform for  $\gamma_A(s)$ ,  $A = [a_1, \dots, a_d]$ .  
 initialize for  $G$  attached to  $A$  `gammamellininivit(A)`  
 $G^{(k)}(t)$  `gammamellininiv(G, t, {k = 0})`  
 asymp. expansion of  $G^{(k)}(t)$  `gammamellininvasymp(A, n, {k = 0})`

## Hypergeometric motives (HGM)

### Hypergeometric templates

Below,  $H$  denotes an hypergeometric template from `hgminit`.  
 HGM template from  $A = (\alpha_j)$ ,  $B = (\beta_k)$  `hgminit(A, {B})`  
 ... from cyclotomic parameters  $D, E$  `hgminit(D, {E})`  
 ... from gamma vector `hgminit(G)`  
 $\alpha$  and  $\beta$  parameters for  $H$  `hgmalph(H)`  
 cyclotomic parameters  $(D, E)$  of  $H$  `hgmcyclo(H)`  
 ... for all  $H$  of degree  $n$  `hgmbdegree(n)`  
 gamma vector for  $H$  `hgmgamma(H)`  
 twist  $A$  and  $B$  by  $1/2$  `hgmtwist(H)`  
 is  $H$  symmetrical at  $t = 1$ ? `hgmissymmetrical(H)`  
 parameters  $[d, w, [P, T], M]$  for  $H$  `hgmparams(H)`

### L-function

Let  $L$  be the  $L$ -function attached to the hypergeometric motive  $(H, t)$ .  
 coefficient  $a_n$  of  $L$  `hgcoef(H, t, n)`  
 coefficients  $[a_1, \dots, a_n]$  of  $L$  `hgcoef(H, t, n)`  
 Euler factor at  $p$  `hgmeulerfactor(H, t, p)`  
 ... and valuation of local conductor `hgmeulerfactor(H, t, p, &e)`  
 return  $L$  as an `Ldata` `lfunhgm(H, t)`

Based on an earlier version by Joseph H. Silverman  
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