

# Algebraic Number Theory

(PARI-GP version 2.18.0)

## Binary Quadratic Forms

create  $ax^2 + bxy + cy^2$  **Qfb**( $a, b, c$ ) or **Qfb**( $[a, b, c]$ )  
 reduce  $x$  ( $s = \sqrt{D}$ ,  $l = \lfloor s \rfloor$ ) **qfbred**( $x, \{flag\}, \{D\}, \{l\}, \{s\}$ )  
 return  $[y, g]$ ,  $g \in \text{SL}_2(\mathbf{Z})$ ,  $y = g \cdot x$  reduced **qfbreds12**( $x$ )  
 composition of forms  $x*y$  or **qfbnucomp**( $x, y, l$ )  
 $n$ -th power of form  $x^n$  or **qfbnupow**( $x, n$ )  
 composition **qfbcomp**( $x, y$ )  
 ... without reduction **qfbcomprow**( $x, y$ )  
 $n$ -th power **qfbpow**( $x, n$ )  
 ... without reduction **qfbpowrow**( $x, n$ )  
 prime form of disc.  $x$  above prime  $p$  **qfbprimeform**( $x, p$ )  
 class number of disc.  $x$  **qfbclassno**( $x$ )  
 Hurwitz class number of disc.  $x$  **qfbhclassno**( $x$ )  
 solve  $Q(x, y) = n$  in integers **qfbsolve**( $Q, n$ )  
 solve  $x^2 + Dy^2 = p$ ,  $p$  prime **qfbcornacchia**( $D, p$ )  
 ...  $x^2 + Dy^2 = 4p$ ,  $p$  prime **qfbcornacchia**( $D, 4 * p$ )

## Quadratic Fields

quadratic number  $\omega = \sqrt{x}$  or  $(1 + \sqrt{x})/2$  **quadgen**( $x$ )  
 minimal polynomial of  $\omega$  **quadpoly**( $x$ )  
 discriminant of  $\mathbf{Q}(\sqrt{x})$  **quaddisc**( $x$ )  
 regulator of real quadratic field **quadregulator**( $x$ )  
 fundamental unit in  $O_D$ ,  $D > 0$  **quadunit**( $D, \{t\}$ )  
 norm of fundamental unit in  $O_D$  **quadunitnorm**( $D$ )  
 index of  $O_{Df^2}^\times$  in  $O_D^\times$  **quadunitindex**( $D, f$ )  
 class group of  $\mathbf{Q}(\sqrt{D})$  **quadclassunit**( $D, \{flag\}, \{t\}$ )  
 Hilbert class field of  $\mathbf{Q}(\sqrt{D})$  **quadhilbert**( $D, \{flag\}$ )  
 ... using specific class invariant ( $D < 0$ ) **polclass**( $D, \{inv\}$ )  
 test if  $T$  is **polclass**( $D$ ); if so return  $D$  **polisclass**( $T$ )  
 ray class field modulo  $f$  of  $\mathbf{Q}(\sqrt{D})$  **quadray**( $D, f, \{flag\}$ )

## General Number Fields: Initializations

The number field  $K = \mathbf{Q}[X]/(f)$  is given by irreducible  $f \in \mathbf{Q}[X]$ .  
 We denote  $\theta = \bar{X}$  the canonical root of  $f$  in  $K$ . A  $nf$  structure  
 contains a maximal order and allows operations on elements and  
 ideals. A  $bnf$  adds class group and units. A  $bnr$  is attached to ray  
 class groups and class field theory. A  $rmf$  is attached to relative  
 extensions  $L/K$ .

init number field structure  $nf$  **nfinit**( $f, \{flag\}$ )  
 known integer basis  $B$  **nfinit**( $[f, B]$ )  
 order maximal at  $vp = [p_1, \dots, p_k]$  **nfinit**( $[f, vp]$ )  
 order maximal at all  $p \leq P$  **nfinit**( $[f, P]$ )  
 certify maximal order **nfcertify**( $nf$ )

### nf members:

a monic  $F \in \mathbf{Z}[X]$  defining  $K$  **nf.pol**  
 number of real/complex places **nf.r1/r2/sign**  
 discriminant of  $nf$  **nf.disc**  
 primes ramified in  $nf$  **nf.p**  
 $T_2$  matrix **nf.t2**  
 complex roots of  $F$  **nf.roots**  
 integral basis of  $\mathbf{Z}_K$  as powers of  $\theta$  **nf.zk**  
 different/codifferent **nf.diff**, **nf.codiff**  
 index  $[\mathbf{Z}_K : \mathbf{Z}[X]/(F)]$  **nf.index**  
 recompute  $nf$  using current precision **nfnewprec**( $nf$ )  
 init relative  $rmf$   $L = K[Y]/(g)$  **rnfinit**( $nf, g$ )  
 init  $bnf$  structure **bnfinit**( $f, 1$ )

**bnf members:** same as  $nf$ , plus

underlying  $nf$   
 class group, regulator  
 fundamental/torsion units  
 add  $S$ -class group and units, yield  $bnfS$   
 init class field structure  $bnr$   
**bnr members:** same as  $bnf$ , plus  
 underlying  $bnf$   
 big ideal structure  
 modulus  $m$   
 structure of  $(\mathbf{Z}_K/m)^*$

**bnf.nf**  
**bnf.clgp**, **bnf.reg**  
**bnf.fu**, **bnf.tu**  
**bnfsunit**( $bnf, S$ )  
**bnrinit**( $bnf, m, \{flag\}$ )  
**bnr.bnf**  
**bnr.bid**  
**bnr.mod**  
**bnr.zkst**

## Fields, subfields, embeddings

### Defining polynomials, embeddings

(some) number fields with Galois group  $G$  **nflist**( $G$ )  
 ... and  $|\text{disc}(K)| = N$  and  $s$  complex places **nflist**( $G, N, \{s\}$ )  
 ... and  $a \leq |\text{disc}(K)| \leq b$  **nflist**( $G, [a, b], \{s\}$ )  
 smallest poly defining  $f = 0$  (slow) **polredabs**( $f, \{flag\}$ )  
 small poly defining  $f = 0$  (fast) **polredbest**( $f, \{flag\}$ )  
 monic integral  $g = Cf(x/L)$  **poltomonic**( $f, \{\&L\}$ )  
 random Tschirnhausen transform of  $f$  **poltschirnhaus**( $f$ )  
 $\mathbf{Q}[t]/(f) \subset \mathbf{Q}[t]/(g)$  ? Isomorphic? **nfisincl**( $f, g$ ), **nfisisom**  
 reverse polmod  $a = A(t) \bmod T(t)$  **modreverse**( $a$ )  
 compositum of  $\mathbf{Q}[t]/(f)$ ,  $\mathbf{Q}[t]/(g)$  **polcompositum**( $f, g, \{flag\}$ )  
 compositum of  $K[t]/(f)$ ,  $K[t]/(g)$  **nfcompositum**( $nf, f, g, \{flag\}$ )  
 splitting field of  $K$  (degree divides  $d$ ) **nfsplitting**( $nf, \{d\}$ )  
 signs of real embeddings of  $x$  **nfeltsign**( $nf, x, \{pl\}$ )  
 complex embeddings of  $x$  **nfeltembed**( $nf, x, \{pl\}$ )  
 $T \in K[t]$ ,  $\#$  of real roots of  $\sigma(T) \in R[t]$  **nfpolsturm**( $nf, T, \{pl\}$ )  
 absolute Weil height **nfweilheight**( $nf, v$ )

### Subfields, polynomial factorization

subfields (of degree  $d$ ) of  $nf$  **nfsubfields**( $nf, \{d\}$ )  
 maximal subfields of  $nf$  **nfsubfieldsmax**( $nf$ )  
 maximal CM subfield of  $nf$  **nfsubfieldscm**( $nf$ )  
 $K_d \subset \mathbf{Q}(\zeta_n)$ , using Gaussian periods **polsubcyclo**( $n, d, \{v\}$ )  
 ... using class field theory **polsubcyclofast**( $n, d$ )  
 roots of unity in  $nf$  **nfroots0f1**( $nf$ )  
 roots of  $g$  belonging to  $nf$  **nfroots**( $nf, g$ )  
 factor  $g$  in  $nf$  **nfactor**( $nf, g$ )

### Linear and algebraic relations

poly of degree  $\leq k$  with root  $x \in \mathbf{C}$  or  $\mathbf{Q}_p$  **algdep**( $x, k$ )  
 alg. dep. with pol. coeffs for series  $s$  **seralgdep**( $s, x, y$ )  
 diff. dep. with pol. coeffs for series  $s$  **serdiffdep**( $s, x, y$ )  
 small linear rel. on coords of vector  $x$  **lindep**( $x$ )

## Basic Number Field Arithmetic (nf)

Number field elements are **t\_INT**, **t\_FRAC**, **t\_POL**, **t\_POLMOD**, or **t\_COL**  
 (on integral basis  $nf.zk$ ).

### Basic operations

$x + y$   
 $x \times y$   
 $x^n$ ,  $n \in \mathbf{Z}$   
 $x/y$   
 $q = x \setminus y := \text{round}(x/y)$   
 $r = x \% y := x - (q \setminus y)y$   
 ...  $[q, r]$  as above  
 reduce  $x$  modulo ideal  $A$  **nfeltreduce**( $nf, x, A$ )  
 absolute trace  $\text{Tr}_{K/\mathbf{Q}}(x)$  **nfelttrace**( $nf, x$ )  
 absolute norm  $N_{K/\mathbf{Q}}(x)$  **nfeltnorm**( $nf, x$ )

**nfeltadd**( $nf, x, y$ )  
**nfeltmul**( $nf, x, y$ )  
**nfeltpow**( $nf, x, n$ )  
**nfeltdiv**( $nf, x, y$ )  
**nfeltdivu**( $nf, x, y$ )  
**nfeltmod**( $nf, x, y$ )  
**nfeltdivrem**( $nf, x, y$ )  
**nfeltreduce**( $nf, x, A$ )  
**nfelttrace**( $nf, x$ )  
**nfeltnorm**( $nf, x$ )

is  $x$  a square? **nfeltissquare**( $nf, x, \{\&y\}$ )  
 ... an  $n$ -th power? **nfeltispower**( $nf, x, n, \{\&y\}$ )

## Multiplicative structure of $K^*$ ; $K^*/(K^*)^n$

valuation  $v_{\mathfrak{p}}(x)$  **nfeltval**( $nf, x, \mathfrak{p}$ )  
 ... write  $x = \pi^{v_{\mathfrak{p}}(x)}y$  **nfeltval**( $nf, x, \mathfrak{p}, \&y$ )  
 quadratic Hilbert symbol (at  $\mathfrak{p}$ ) **nfhilbert**( $nf, a, b, \{\mathfrak{p}\}$ )  
 $b$  such that  $xb^n = v$  is small **idealredmodpower**( $nf, x, n$ )

## Maximal order and discriminant

integral basis of field  $\mathbf{Q}[x]/(f)$  **nfbasis**( $f$ )  
 field discriminant of  $\mathbf{Q}[x]/(f)$  **nfdisc**( $f$ )  
 ... and factorization **nfdiscfactors**( $f$ )  
 express  $x$  on integer basis **nfalgtobasis**( $nf, x$ )  
 express element  $x$  as a polmod **nfbasistoalg**( $nf, x$ )

## Hecke Grossencharacters

Let  $K$  be a number field and  $m$  a modulus. A  $gchar$  structure  
 describes the group of Hecke Grossencharacters of  $K$  of modulus  $m$   
 and allows computations with these characters. A character  $\chi$  is  
 described by its components modulo  $gc.cyc$ .

init  $gchar$  structure  $gc$  for modulus  $m$  **gcharinit**( $bnf, m, \{cm\}$ )

### gc members:

underlying  $bnf$  **gc.bnf**  
 modulus **gc.mod**  
 elementary divisors (including 0s) **gc.cyc**  
 recompute  $gc$  using current precision **gcharnewprec**( $gc$ )  
 evaluate Hecke character  $chi$  at ideal  $id$  **gchareval**( $gc, chi, id$ )  
 exponent column of  $id$  in  $\mathbf{R}^n$  **gcharideallog**( $gc, id$ )  
 log representation of ideal  $id$  **gcharlog**( $gc, id$ )  
 ... of character  $\chi$  **gcharallog**( $gc, chi$ )  
 exponent vector of  $\chi$  in  $\mathbf{R}^n$  **gcharparameters**( $gc, chi$ )  
 conductor of  $\chi$  **gcharconductor**( $gc, chi$ )  
 L-function of  $\chi$  **lfunccreate**( $[gc, chi]$ )  
 local component  $\chi_v$  of  $\chi$  **gcharlocal**( $gc, chi, v$ )  
 $\chi$  s.t.  $\chi_v \approx Lchiv[i]$  for  $v = Lv[i]$  **gcharidentify**( $gc, Lv, Lchiv$ )  
 basis of group of algebraic characters **gcharalgebraic**( $gc$ )  
 algebraic character of given infinity type **gcharalgebraic**( $gc, type$ )  
 is  $\chi$  algebraic? **gcharisalgebraic**( $gc, chi$ )

## Dedekind Zeta Function $\zeta_K$ , Hecke $L$ series

$R = [c, w, h]$  in initialization means we restrict  $s \in \mathbf{C}$  to domain  
 $|\Re(s) - c| < w$ ,  $|\Im(s)| < h$ ;  $R = [w, h]$  encodes  $[1/2, w, h]$  and  $[h]$   
 encodes  $R = [1/2, 0, h]$  (critical line up to height  $h$ ).

$\zeta_K$  as Dirichlet series,  $N(I) \leq b$  **dirzetak**( $nf, b$ )  
 init  $\zeta_K^{(k)}(s)$  for  $k \leq n$  **L = lfunit**( $bnf, R, \{n = 0\}$ )  
 compute  $\zeta_K(s)$  ( $n$ -th derivative) **lfun**( $L, s, \{n = 0\}$ )  
 compute  $\Lambda_K(s)$  ( $n$ -th derivative) **lfunlambda**( $L, s, \{n = 0\}$ )

init  $L_K^{(k)}(s, \chi)$  for  $k \leq n$  **L = lfunit**( $[bnr, chi], R, \{n = 0\}$ )  
 compute  $L_K(s, \chi)$  ( $n$ -th derivative) **lfun**( $L, s, \{n\}$ )  
 Artin root number of  $K$  **bnrrootnumber**( $bnr, chi, \{flag\}$ )  
 $L(1, \chi)$ , for all  $\chi$  trivial on  $H$  **bnrL1**( $bnr, \{H\}, \{flag\}$ )

## Class Groups & Units (bnf, bnr)

Class field theory data  $a_1, \{a_2\}$  is usually  $bnr$  (ray class field),  
 $bnr, H$  (congruence subgroup) or  $bnr, \chi$  (character on **bnr.clgp**).  
 Any of these define a unique abelian extension of  $K$ .

units /  $S$ -units **bnfunits**( $bnf, \{S\}$ )  
 remove GRH assumption from  $bnf$  **bnfcertify**( $bnf$ )

# Algebraic Number Theory

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expo. of ideal  $x$  on class gp      `bnfisprincipal(bnf, x, {flag})`  
 ... on ray class gp                `bnrisprincipal(bnr, x, {flag})`  
 expo. of  $x$  on fund. units        `bnfisunit(bnf, x)`  
 ... on  $S$ -units,  $U$  is `bnfunits(bnf, S)`      `bnfisunit(bnfs, x, U)`  
 signs of real embeddings of  $bnf$ .fu      `bnfsignunit(bnf)`  
 narrow class group                `bnfnarrow(bnf)`

## Class Field Theory

ray class number for modulus  $m$         `bnrclassno(bnf, m)`  
 discriminant of class field            `bnrdisc(a1, {a2})`  
 ray class numbers,  $l$  list of moduli      `bnrclasslist(bnf, l)`  
 discriminants of class fields      `bnrdisclist(bnf, l, {arch}, {flag})`  
 decode output from `bnrdisclist`        `bnfdecodemodule(nf, fa)`  
 is modulus the conductor?            `bnrisconductor(a1, {a2})`  
 is class field  $(bnr, H)$  Galois over  $K^G$       `bnrisgalois(bnr, G, H)`  
 action of automorphism on `bnr.gen`      `bnrgaloismatrix(bnr, aut)`  
 apply `bnrgaloismatrix M` to  $H$       `bnrgaloisapply(bnr, M, H)`  
 characters on `bnr.clgp` s.t.  $\chi(g_i) = e(v_i)$       `bnrchar(bnr, g, {v})`  
 conductor of character  $\chi$             `bnrconductor(bnr, chi)`  
 conductor of extension                `bnrconductor(a1, {a2}, {flag})`  
 conductor of extension  $K[Y]/(g)$         `rnfconductor(bnf, g)`  
 canonical projection  $Cl_F \rightarrow Cl_f, f | F$       `bnrmap`  
 Artin group of extension  $K[Y]/(g)$         `rnfnormgroup(bnr, g)`  
 subgroups of  $bnr$ , index  $\leq b$         `subgrouplist(bnr, b, {flag})`  
 compositum as `[bnr, H]`            `bnrcompositum([bnr1, H1], [bnr2, H2])`  
 class field defined by  $H < Cl_f$         `bnrclassfield(bnr, H)`  
 ... low level equivalent, prime degree      `rnfkummer(bnr, H)`  
 same, using Stark units (real field)      `bnrstark(bnr, {sub}, {flag})`  
 Stark unit                            `bnrstarkunit(bnr, {sub})`  
 is  $a$  an  $n$ -th power in  $K_v$ ?            `nfislocalpower(nf, v, a, n)`  
 cyclic  $L/K$  satisf. local conditions      `nfgrunwaldwang(nf, P, D, pl)`

## Cyclotomic and Abelian fields theory

An Abelian field  $F$  given by a subgroup  $H < (Z/fZ)^*$  is described by an argument  $F$ , e.g.  $f$  (for  $H = 1$ , i.e.  $Q(\zeta_f)$ ) or  $[G, H]$ , where  $G$  is `idealstar(f, 1)`, or a minimal polynomial.  
 minus class number  $h^-(F)$             `subcyclohminus(F)`  
 ...  $p$ -part                            `subcyclohminus(F, p)`  
 minus part of Iwasawa polynomials      `subcycloiwawasawa(F, p)`  
 $p$ -Sylow of  $Cl(F)$                     `subcycloplgp(F, p)`

## Logarithmic class group

logarithmic  $\ell$ -class group            `bnflog(bnf, \ell)`  
 $[\tilde{e}(F_v/Q_p), \tilde{f}(F_v/Q_p)]$             `bnfloget(bnf, pr)`  
 exp deg  $F(A)$                         `bnflogdegree(bnf, A, \ell)`  
 is  $\ell$ -extension  $L/K$  locally cyclotomic      `rnfislocalcyclo(rnf)`

**Ideals:** elements, primes, or matrix of generators in HNF

is  $id$  an ideal in  $nf$ ?                `nfisideal(nf, id)`  
 is  $x$  principal in  $bnf$ ?                `bnfisprincipal(bnf, x)`  
 give  $[a, b]$ , s.t.  $aZ_K + bZ_K = x$       `idealtwoelt(nf, x, {a})`  
 put ideal  $a(aZ_K + bZ_K)$  in HNF form      `idealhnf(nf, a, {b})`  
 norm of ideal  $x$                         `idealnrm(nf, x)`  
 minimum of ideal  $x$  (direction  $v$ )      `ideallmin(nf, x, v)`  
 LLL-reduce the ideal  $x$  (direction  $v$ )      `idealred(nf, x, {v})`

## Ideal Operations

add ideals  $x$  and  $y$                     `idealadd(nf, x, y)`  
 multiply ideals  $x$  and  $y$                 `idealmul(nf, x, y, {flag})`  
 intersection of ideal  $x$  with  $Q$         `idealdn(nf, x)`  
 intersection of ideals  $x$  and  $y$         `idealintersect(nf, x, y, {flag})`  
 $n$ -th power of ideal  $x$                 `idealpow(nf, x, n, {flag})`  
 inverse of ideal  $x$                         `idealinvt(nf, x)`

divide ideal  $x$  by  $y$                     `idealdiv(nf, x, y, {flag})`  
 Find  $(a, b) \in x \times y, a + b = 1$         `idealaddtoone(nf, x, {y})`  
 coprime integral  $A, B$  such that  $x = A/B$       `idealnumden(nf, x)`

## Primes and Multiplicative Structure

check whether  $x$  is a maximal ideal      `idealismaximal(nf, x)`  
 factor ideal  $x$  in  $Z_K$                 `idealfactor(nf, x)`  
 expand ideal factorization in  $K$         `idealfactorback(nf, f, {e})`  
 is ideal  $A$  an  $n$ -th power?                `idealispower(nf, A, n)`  
 expand elt factorization in  $K$         `nfactorback(nf, f, {e})`  
 decomposition of prime  $p$  in  $Z_K$         `idealprimedec(nf, p)`  
 valuation of  $x$  at prime ideal  $pr$       `idealval(nf, x, pr)`  
 weak approximation theorem in  $nf$       `idealchinese(nf, x, y)`  
 $a \in K$ , s.t.  $v_p(a) = v_p(x)$  if  $v_p(x) \neq 0$       `idealappr(nf, x)`  
 $a \in K$  such that  $(a \cdot x, y) = 1$         `idealcoprime(nf, x, y)`  
 give  $bid$  = structure of  $(Z_K/id)^*$       `idealstar(nf, id, {flag})`  
 structure of  $(1 + \mathfrak{p})/(1 + \mathfrak{p}^k)$       `idealprincipalunits(nf, pr, k)`  
 discrete log of  $x$  in  $(Z_K/bid)^*$       `ideallog(nf, x, bid)`  
`idealstar` of all ideals of norm  $\leq b$       `ideallist(nf, b, {flag})`  
 add Archimedean places                `ideallistarch(nf, b, {ar}, {flag})`  
 init `modpr` structure                `nfmodprinit(nf, pr, {v})`  
 project  $t$  to  $Z_K/pr$                 `nfmodpr(nf, t, modpr)`  
 lift from  $Z_K/pr$                         `nfmodprlift(nf, t, modpr)`

## Galois theory over Q

conjugates of a root  $\theta$  of  $nf$             `nfgaloisconj(nf, {flag})`  
 apply Galois automorphism  $s$  to  $x$       `nfgaloisapply(nf, s, x)`  
 Galois group of field  $Q[x]/(f)$         `polgalois(f)`  
 resolvent field of  $Q[x]/(f)$             `nfresolvent(f)`  
 initializes a Galois group structure  $G$       `galoisinit(pol, {den})`  
 ... for the splitting field of  $pol$       `galoissplittinginit(pol, {d})`  
 character table of  $G$                     `galoischartable(G)`  
 conjugacy classes of  $G$                 `galoisconjclasses(G)`  
 $\det(1 - \rho(g)T)$ ,  $\chi$  character of  $\rho$       `galoischarpoly(G, \chi, {o})`  
 $\det(\rho(g))$ ,  $\chi$  character of  $\rho$         `galoischarpoly(G, \chi, {o})`  
 action of  $p$  in `nfgaloisconj` form      `galoispermtpol(G, {p})`  
 identify as abstract group                `galoisidentify(G)`  
 export a group for GAP/MAGMA          `galoisexport(G, {flag})`  
 subgroups of the Galois group  $G$         `galoissubgroups(G)`  
 is subgroup  $H$  normal?                `galoisisnormal(G, H)`  
 subfields from subgroups                `galoissubfields(G, {flag}, {v})`  
 fixed field                            `galoisfixedfield(G, perm, {flag}, {v})`  
 Frobenius at maximal ideal  $P$         `idealfrobenius(nf, G, P)`  
 ramification groups at  $P$                 `idealramgroups(nf, G, P)`  
 is  $G$  abelian?                        `galoisisabelian(G, {flag})`  
 abelian number fields/ $Q$                 `galoissubcyclo(N, H, {flag}, {v})`

## The galpol package

query the package: polynomial          `galoisgetpol(a, b, {s})`  
 ...: permutation group                `galoisgetgroup(a, b)`  
 ...: group description                `galoisgetname(a, b)`

## Relative Number Fields (rnf)

Extension  $L/K$  is defined by  $T \in K[x]$ .  
 absolute equation of  $L$                 `rnfequation(nf, T, {flag})`  
 is  $L/K$  abelian?                        `rnfisabelian(nf, T)`  
 relative `nfgalgtobasis`                `rnfalgtobasis(rnf, x)`  
 relative `nfbasistoalg`                `rnfbasistoalg(rnf, x)`  
 relative `idealhnf`                        `rnfidealhnf(rnf, x)`

relative `idealmul`                        `rnfidealmul(rnf, x, y)`  
 relative `idealtwoelt`                `rnfidealtwoelt(rnf, x)`

## Lifts and Push-downs

absolute  $\rightarrow$  relative representation for  $x$       `rnfeltabstorel(rnf, x)`  
 relative  $\rightarrow$  absolute representation for  $x$       `rnfeltreltoabs(rnf, x)`  
 lift  $x$  to the relative field            `rnfeltup(rnf, x)`  
 push  $x$  down to the base field        `rnfeltdown(rnf, x)`  
 idem for  $x$  ideal: `(rnfideal)reltoabs, abstorel, up, down`

## Norms and Trace

relative norm of element  $x \in L$         `rnfeltnorm(rnf, x)`  
 relative trace of element  $x \in L$         `rnfeltrace(rnf, x)`  
 absolute norm of ideal  $x$                 `rnfidealnrmabs(rnf, x)`  
 relative norm of ideal  $x$                 `rnfidealnrmrel(rnf, x)`  
 solutions of  $N_{K/Q}(y) = x \in Z$         `bnfisintnorm(bnf, x)`  
 is  $x \in Q$  a norm from  $K$ ?                `bnfisnorm(bnf, x, {flag})`  
 initialize  $T$  for norm eq. solver        `rnfnorminit(K, pol, {flag})`  
 is  $a \in K$  a norm from  $L$ ?                `rnfnorm(T, a, {flag})`  
 initialize  $t$  for Thue equation solver      `thueinit(f)`  
 solve Thue equation  $f(x, y) = a$         `thue(t, a, {sol})`  
 characteristic poly. of  $a$  mod  $T$         `rnfcharpoly(nf, T, a, {v})`

## Factorization

factor ideal  $x$  in  $L$                     `rnfidealfactor(rnf, x)`  
 $[S, T]: T_{i,j} | S_i; S$  primes of  $K$  above  $p$       `rnfidealprimedec(rnf, p)`

## Maximal order $Z_L$ as a $Z_K$ -module

relative `polredbest`                    `rnfpolredbest(nf, T)`  
 relative `polredabs`                    `rnfpolredabs(nf, T)`  
 relative Dedekind criterion, prime  $pr$       `rnfdedekind(nf, T, pr)`  
 discriminant of relative extension        `rnfdisc(nf, T)`  
 pseudo-basis of  $Z_L$                     `rnfpsudobasis(nf, T)`

**General  $Z_K$ -modules:**  $M = [\text{matrix, vec. of ideals}] \subset L$   
 relative HNF / SNF                    `nfhnf(nf, M), nfnfnf`  
 multiple of  $\det M$                     `nfdetint(nf, M)`  
 HNF of  $M$  where  $d = nfdetint(M)$       `nfhnfmod(x, d)`  
 reduced basis for  $M$                     `rnfllgram(nf, T, M)`  
 determinant of pseudo-matrix  $M$         `rnfdet(nf, M)`  
 Steinitz class of  $M$                     `rnfsteinitz(nf, M)`  
 $Z_K$ -basis of  $M$  if  $Z_K$ -free, or 0        `rnfhnfbasis(bnf, M)`  
 $n$ -basis of  $M$ , or  $(n + 1)$ -generating set      `rnfbasis(bnf, M)`  
 is  $M$  a free  $Z_K$ -module?                `rnfisfree(bnf, M)`

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## Associative Algebras

$A$  is a general associative algebra given by a multiplication table  $mt$  (over  $\mathbf{Q}$  or  $\mathbf{F}_p$ ); represented by  $al$  from `algtableinit`.

create  $al$  from  $mt$  (over  $\mathbf{F}_p$ )      `algtableinit(mt, {p = 0})`  
 group algebra  $\mathbf{Q}[G]$  (or  $\mathbf{F}_p[G]$ )      `algggroup(G, {p = 0})`  
 center of group algebra      `algggroupcenter(G, {p = 0})`

### Properties

is  $(mt, p)$  OK for `algtableinit`?      `algisassociative(mt, {p = 0})`  
 multiplication table  $mt$       `algmtable(al)`  
 dimension of  $A$  over prime subfield      `algdim(al)`  
 characteristic of  $A$       `algchar(al)`  
 is  $A$  commutative?      `algiscommutative(al)`  
 is  $A$  simple?      `algissimple(al)`  
 is  $A$  semi-simple?      `algissemisimple(al)`  
 center of  $A$       `algcenter(al)`  
 Jacobson radical of  $A$       `algradical(al)`  
 radical  $J$  and simple factors of  $A/J$       `algsimpledec(al)`

### Operations on algebras

create  $A/I$ ,  $I$  two-sided ideal      `algquotient(al, I)`  
 create  $A_1 \otimes A_2$       `algtensor(al1, al2)`  
 create subalgebra from basis  $B$       `algsubalg(al, B)`  
 quotients by ortho. central idempotents  $e$       `algcentralproj(al, e)`  
 isomorphic alg. with integral mult. table      `algmakeintegral(mt)`  
 prime subalgebra of semi-simple  $A$  over  $\mathbf{F}_p$       `algprimesubalg(al)`  
 find isomorphism  $A \cong M_d(\mathbf{F}_q)$       `algsplit(al)`

### Operations on lattices in algebras

lattice generated by cols. of  $M$       `alglathnf(al, M)`  
 $\dots$  by the products  $xy$ ,  $x \in lat1$ ,  $y \in lat2$       `alglatmul(al, lat1, lat2)`  
 sum  $lat1 + lat2$  of the lattices      `alglatadd(al, lat1, lat2)`  
 intersection  $lat1 \cap lat2$       `alglatinter(al, lat1, lat2)`  
 test  $lat1 \subset lat2$       `alglatsubset(al, lat1, lat2)`  
 generalized index  $(lat2 : lat1)$       `alglatindex(al, lat1, lat2)`  
 $\{x \in al \mid x \cdot lat1 \subset lat2\}$       `alglatlefttransporter(al, lat1, lat2)`  
 $\{x \in al \mid lat1 \cdot x \subset lat2\}$       `alglatrighttransporter(al, lat1, lat2)`  
 test  $x \in lat$  (set  $c = \text{coord. of } x$ )      `alglatcontains(al, lat, x, {\&c})`  
 element of  $lat$  with coordinates  $c$       `alglatelement(al, lat, c)`

### Operations on elements

$a + b$ ,  $a - b$ ,  $-a$       `algadd(al, a, b)`, `algsub`, `algneg`  
 $a \times b$ ,  $a^2$       `algmul(al, a, b)`, `algsqr`  
 $a^n$ ,  $a^{-1}$       `algpow(al, a, n)`, `alginv`  
 is  $x$  invertible? (then set  $z = x^{-1}$ )      `algisinv(al, x, {\&z})`  
 find  $z$  such that  $x \times z = y$       `algdivl(al, x, y)`  
 find  $z$  such that  $z \times x = y$       `algdivr(al, x, y)`  
 does  $z$  s.t.  $x \times z = y$  exist? (set it)      `algisdivl(al, x, y, {\&z})`  
 matrix of  $v \mapsto x \cdot v$       `algtomatrix(al, x)`  
 absolute norm      `algnorm(al, x)`  
 absolute trace      `algtrace(al, x)`  
 absolute char. polynomial      `algcharpoly(al, x)`  
 given  $a \in A$  and polynomial  $T$ , return  $T(a)$       `algpoleval(al, T, a)`  
 random element in a box      `algrandom(al, b)`

## Central Simple Algebras

$A$  is a central simple algebra over a number field  $K$ ; represented by  $al$  from `alginit`;  $K$  is given by a  $nf$  structure.

create CSA from data      `alginit(B, C, {v}, {flag = 3})`  
 multiplication table over  $K$        $B = K$ ,  $C = mt$   
 cyclic algebra  $(L/K, \sigma, b)$        $B = rnf$ ,  $C = [sigma, b]$   
 quaternion algebra  $(a, b)_K$        $B = K$ ,  $C = [a, b]$   
 matrix algebra  $M_d(K)$        $B = K$ ,  $C = d$   
 local Hasse invariants over  $K$        $B = K$ ,  $C = [d, [PR, HF], HI]$   
 ramification set of quaternion algebra       $B = K$ ,  $C = [PR, HI]$   
 Hamilton quaternion algebra  $(-1, -1)_\mathbf{R}$        $B = 1.$ ,  $C = 1/2$   
 recover  $(a, b)$  if  $al = (a, b)_K$       `algisquatalg(al)`  
 recompute  $al$  using current precision      `algnewprec(al)`

### Properties

type of  $al$  ( $mt$ , CSA)      `algtype(al)`  
 dimension of  $A$  over  $\mathbf{Q}$       `algdim(al, 1)`  
 dimension of  $al$  over its center  $K$       `algdim(al)`  
 degree of  $A$  ( $= \sqrt{\dim_K A}$ )      `algdegree(al)`  
 $al$  a cyclic algebra  $(L/K, \sigma, b)$ ; return  $\sigma$       `algaut(al)`  
 $\dots$  return  $b$       `algb(al)  
 $\dots$  return  $L/K$ , as an  $rnf$       algsplittingfield(al)  
 split  $A$  over an extension of  $K$       algsplittingdata(al)  
 splitting field of  $A$  as an  $rnf$  over center      algsplittingfield(al)  
 multiplication table over center      algrelmtable(al)  
 places of  $K$  at which  $A$  ramifies      algramifiedplaces(al)  
 Hasse invariants at finite places of  $K$       alghassef(al)  
 Hasse invariants at infinite places of  $K$       alghassei(al)  
 Hasse invariant at place  $v$       alghasse(al, v)  
 index of  $A$  over  $K$  (at place  $v$ )      algindex(al, {v})  
 is  $al$  a division algebra? (at place  $v$ )      algisdivision(al, {v})  
 is  $A$  ramified? (at place  $v$ )      algisramified(al, {v})  
 is  $A$  split? (at place  $v$ )      algissplit(al, {v})`

### Operations on elements

$x \in nf$  as an element of  $al$       `algeltfromnf(al, x)`  
 reduced norm      `algnorm(al, x)`  
 reduced trace      `algtrace(al, x)`  
 reduced char. polynomial      `algcharpoly(al, x)`  
 express  $x$  on integral basis      `algalgtobasis(al, x)`  
 convert  $x$  to algebraic form      `algbasistoalg(al, x)`  
 express quaternion  $x$  on integral basis      `algquattobasis(al, x)`  
 convert  $x$  to quaternion form      `algbasistoquat(al, x)`  
 quaternion conjugate of  $x$       `alginvol(al, x)`  
 map  $x \in A$  to  $M_d(L)$ ,  $L$  split. field      `algtomatrix(al, x)`  
 init mod  $pr$  map  $\mathcal{O}_0 \rightarrow M_k(\mathbf{F}_q)$       `algmodpinit(al, pr, {v})`  
 project  $x$  to  $M_k(\mathbf{F}_q)$       `algmodpr(al, x, modP)`  
 lift from  $M_k(\mathbf{F}_q)$  to  $\mathcal{O}_0$       `algmodprlift(al, x, modP)`

### Orders

$\mathbf{Z}$ -basis of order  $\mathcal{O}_0$       `algbasis(al)`  
 discriminant of order  $\mathcal{O}_0$       `algdisc(al)`  
 $\mathbf{Z}$ -basis of natural order in terms  $\mathcal{O}_0$ 's basis      `alginvbasis(al)`  
 $\mathbf{Z}$ -basis of Eichler order of level  $N$       `algeichlerbasis(al, N)`

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