

Elliptic Curves

(PARI-GP version 2.10.0)

An elliptic curve is initially given by 5-tuple $v = [a_1, a_2, a_3, a_4, a_6]$ attached to Weierstrass model or simply $[a_4, a_6]$. It must be converted to an *ell* struct.

Initialize *ell* struct over domain D **E** = **ellinit**($v, \{D = 1\}$)
over **Q** $D = 1$
over **F_p** $D = p$
over **F_q**, $q = p^f$ $D = \text{ffgen}([p, f])$
over **Q_p**, precision n $D = O(p^n)$
over **C**, current bitprecision $D = 1.0$
over number field K $D = nf$

Points are $[x, y]$, the origin is $[0]$. Struct members accessed as **E.member**:

- All domains: **E.a1, a2, a3, a4, a6, b2, b4, b6, b8, c4, c6, disc, j**
- E defined over **R** or **C**
 x -coords. of points of order 2 **E.roots**
periods / quasi-periods **E.omega, E.eta**
volume of complex lattice **E.area**

- E defined over **Q_p**
residual characteristic **E.p**
If $|j|_p > 1$: Tate's $[u^2, u, q, [a, b], \mathcal{L}]$ **E.tate**

- E defined over **F_q**
characteristic **E.p**
 $\#E(\mathbf{F}_q)/\text{cyclic structure/generators}$ **E.no, E.cyc, E.gen**

- E defined over **Q**
generators of $E(\mathbf{Q})$ (require **elldata**) **E.gen**
 $[a_1, a_2, a_3, a_4, a_6]$ from j -invariant **ellfromj(j)**
cubic/quartic/biquadratic to Weierstrass **ellfromeqn(eq)**
add points $P + Q / P - Q$ **elladd(E, P, Q), ellsub**
negate point **ellneg(E, P)**
compute $n \cdot z$ **ellmul(E, z, n)**
check if z is on E **ellisoncurve(E, z)**
order of torsion point z **ellorder(E, z)**
 y -coordinates of point(s) for x **ellordinate(E, x)**
point $[\varphi(z), \varphi'(z)]$ corresp. to z **ellztopoint(E, z)**
complex z such that $p = [\varphi(z), \varphi'(z)]$ **ellpointtoz(E, p)**

- **Change of Weierstrass models, using** $v = [u, r, s, t]$
change curve E using v **ellchangecurve(E, v)**
change point z using v **ellchangept(z, v)**
change point z using inverse of v **ellchangeptinv(z, v)**

- **Twists and isogenies**
quadratic twist **elltwist(E, D)**
 n -division polynomial $f_n(x)$ **elldivpol(E, n, \{x\})**
 $[n]P = (\phi_n \psi_n : \omega_n : \psi_n^3)$; return (ϕ_n, ψ_n^2) **ellxn(E, n, v)**
isogeny from E to E/G **ellisogeny(E, G)**
apply isogeny to g (point or isogeny) **ellisogenyapply(f, g)**

- **Formal group**
formal exponential, n terms **ellformalexp(E, \{n\}, \{v\})**
formal logarithm, n terms **ellformallog(E, \{n\}, \{v\})**
 $L(-x/y) \in \mathbf{Q}_p$; $P \in E(\mathbf{Q}_p)$ **ellpadiclog(E, p, n, P)**
 $[x, y]$ in the formal group **ellformalpoint(E, \{n\}, \{v\})**
 $[f, g], \omega = f(t)dt, x\omega = g(t)dt$ **ellformaldifferential**
 $w = -1/y$ in parameter $-x/y$ **ellformalw(E, \{n\}, \{v\})**

Curves over finite fields, Pairings

random point on E **random(E)**
 $\#E(\mathbf{F}_q)$ **ellcard(E)**
 $\#E(\mathbf{F}_q)$ with almost prime order **ellsea(E, \{tors\})**
structure $\mathbf{Z}/d_1\mathbf{Z} \times \mathbf{Z}/d_2\mathbf{Z}$ of $E(\mathbf{F}_q)$ **ellgroup(E)**
is E supersingular? **ellissupersingular(E)**
Weil pairing of m -torsion pts x, y **ellweilpairing(E, x, y, m)**
Tate pairing of x, y ; x m -torsion **elltatepairing(E, x, y, m)**
Discrete log, find n s.t. $P = [n]Q$ **elllog(E, P, Q, \{ord\})**

Curves over Q

- **Reduction, minimal model**
minimal model of E/\mathbf{Q} **ellminimalmodel(E, \{\&v\})**
quadratic twist of minimal conductor **ellminimaltwist**
multiple with good reduction **ellnonsingularmultiple(E, P)**

- **Complex heights**
canonical height of P **ellheight(E, P)**
canonical bilinear form taken at P, Q **ellheight(E, P, Q)**
height regulator matrix for pts in x **ellheightmatrix(E, x)**

- **p -adic heights**
cyclotomic p -adic height of $P \in E(\mathbf{Q})$ **ellpadicheight(E, P, n)**
... bilinear form at $P, Q \in E(\mathbf{Q})$ **ellpadicheight(E, P, n, Q)**
... matrix at vector of points **ellpadicheightmatrix(E, p, n, x)**
Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1(E/\mathbf{Q})$ **ellpadicfrobenius(E, p, n)**
slope of unit eigenvector of Frobenius **ellpads2(E, p, n)**

- **Isogenous curves**
matrix of isogeny degrees for \mathbf{Q} -isog. curves **ellisomat(E)**
a modular equation of prime degree N **ellmodulareqn(N)**

- **L -function**
 p -th coeff a_p of L -function, p prime **ellap(E, p)**
 E supersingular at $p?$ **ellissupersingular(E, p)**
 k -th coeff a_k of L -function **ellak(E, k)**
 $L(E, s)$ (using less memory than **lfun**) **elllseries(E, s)**
 $L^{(r)}(E, 1)$ (using less memory than **lfun**) **elll1(E, r)**
a Heegner point on E of rank 1 **ellheegner(E)**
order of vanishing at 1 **ellanalyticrank(E, \{eps\})**
root number for $L(E, \cdot)$ at p **ellrootno(E, \{p\})**
modular parametrization of E **elltaniyama(E)**
degree of modular parametrization **ellmoddegree(E)**
 p -adic L -function of E at χ^s **ellpadicL(E, p, n, \{s = 0\})**

- **Elldata package, Cremona's database:**
db code "11a1" \leftrightarrow [*conductor, class, index*] **ellconvertname(s)**
generators of Mordell-Weil group **ellgenerators(E)**
look up E in database **ellidentify(E)**
all curves matching criterion **ellsearch(N)**
loop over curves with cond. from a to b **forell(E, a, b, seq)**

Curves over number field K

- coeff a_p of L -function **ellap(E, p)**
Kodaira type of \mathfrak{p} -fiber of E **elllocalred(E, p)**
integral model of E/K **ellintegralmodel(E, \{\&v\})**
minimal model of E/K **ellminimalmodel(E, \{\&v\})**
cond, min mod, Tamagawa num $[N, v, c]$ **ellglobalred(E)**
 $P \in E(K)$ n -divisible? $[n]Q = P$ **ellisdivisible(E, P, n, \{\&Q\})**

L -function

A domain $D = [c, w, h]$ in initialization mean we restrict $s \in \mathbf{C}$ to domain $|\Re(s) - c| < w, |\Im(s)| < h$; $D = [w, h]$ encodes $[1/2, w, h]$ and $[h]$ encodes $D = [1/2, 0, h]$ (critical line up to height h).
vector of first n a_k 's in L -function **ellan(E, n)**
init $L^{(k)}(E, s)$ for $k \leq n$ **L = lfuninit(E, D, \{n = 0\})**
compute $L(E, s)$ (n -th derivative) **lfun(L, s, \{n = 0\})**
torsion subgroup with generators **elltors(E)**

Other curves of small genus

A hyperelliptic curve is given by a pair $[P, Q]$ ($y^2 + Qy = P$ with $Q^2 + 4P$ squarefree) or a single squarefree polynomial P ($y^2 = P$).
reduction of $y^2 + Qy = P$ (genus 2) **genus2red([P, Q], \{p\})**
find a rational point on a conic, ${}^t xGx = 0$ **qfsolve(G)**
quadratic Hilbert symbol (at p) **hilbert(x, y, \{p\})**
all solutions in \mathbf{Q}^3 of ternary form **qfparam(G, x)**
 $P, Q \in \mathbf{F}_q[X]$; char. poly. of Frobenius **hyperellcharpoly([P, Q])**
matrix of Frobenius on $\mathbf{Q}_p \otimes H_{dR}^1$ **hyperellpadicfrobenius**

Elliptic & Modular Functions

$w = [\omega_1, \omega_2]$ or *ell* struct (**E.omega**), $\tau = \omega_1/\omega_2$.
arithmetic-geometric mean **agm(x, y)**
elliptic j -function $1/q + 744 + \dots$ **ellj(x)**
Weierstrass $\sigma/\wp/\zeta$ function **ellsigma(w, z), ellwp, ellzeta**
periods/quasi-periods **ellperiods(E, \{flag\}), elleta(w)**
 $(2i\pi/\omega_2)^k E_k(\tau)$ **elleisnum(w, k, \{flag\})**
modified Dedekind η func. $\prod(1 - q^n)$ **eta(x, \{flag\})**
Dedekind sum $s(h, k)$ **sumdedekind(h, k)**
Jacobi sine theta function **theta(q, z)**
 k -th derivative at $z=0$ of $\theta(q, z)$ **thetanullk(q, k)**
Weber's f functions **weber(x, \{flag\})**
modular pol. of level N **polmodular(N, \{inv = j\})**
Hilbert class polynomial for $\mathbf{Q}(\sqrt{D})$ **polclass(D, \{inv = j\})**

Based on an earlier version by Joseph H. Silverman
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